Standard Model and QCD

Problem Sheet 10

2 July 2024

Chiral symmetry breaking and sigma models

At the classical level the Lagrangian of QCD with two massless quarks (u and d) is invariant under the global symmetry group

$$SU(2)_L \otimes SU(2)_R \otimes U(1)_A \otimes U(1)_V \tag{1}$$

In this exercise we focus on the global chiral symmetries $SU(2)_L$ and $SU(2)_R$.

a) Write down the QCD Lagrangian for two massless quarks and identify the global chiral symmetries $SU(2)_L$ and $SU(2)_R$ under which it is invariant. How would this symmetry group generalize if we add more massless quarks to the theory ?

b) Argue that a condensate

$$< \bar{u}u > = < \bar{d}d > = v^3 \tag{2}$$

breaks $SU(2)_L \otimes SU(2)_R$ down to its diagonal subgroup $SU(2)_V$. How many Goldstone bosons emerge ? How many Goldstone bosons emerge in the case of three massless quarks when

$$SU(3)_L \otimes SU(3)_R \longrightarrow SU(3)_V\,? \tag{3}$$

c) In particle physics, we often use phenomenological so-called "sigma model Lagrangians" in order to describe the dynamics of the (pseudo-)Goldstone bosons at low energies. We shall now investigate Gell-Mann and Levy linear and non-linear sigma models which correspond to the QCD Lagrangian with two massless quarks.

Consider the Lagrangian density of the linear sigma model with the four scalar fields $\phi^{i\bar{j}}$ transforming linearly under $SU(2)_L \otimes SU(2)_R$,

$$\mathcal{L} = \frac{1}{2}|\partial_\mu \phi|^2 + \frac{\mu^2}{2}|\phi|^2 - \frac{\lambda}{4}|\phi|^4 \tag{4}$$

where $\lambda > 0$ and barred indices belong to $SU(2)_R$. Minimize the potential, take the VEV v of the scalar fields in such a way that the symmetry is broken into the diagonal subrgroup and identify the (pseudo-)Goldstone bosons π^1, π^2 and π^3. Expand the Lagrangian around the VEV by introducing a fluctuation, $\sigma(x)$, and write it down in terms of σ and π^a ($a = 1, 2, 3$).

d) Argue what happens physically to σ in the double scaling limit $\mu^2 \to \infty$, $\lambda \to \infty$, v^2 fixed. Argue that the fields satisfy the constraint $\pi^a\pi^a + \sigma^2 = -2v\sigma$ in this limit.

e) Plugging this constraint into the Lagrangian of the linear sigma model leads to the so-called non-linear sigma model for the three Goldstone bosons π^a

$$\mathcal{L} = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{2}\frac{\pi^a \partial_\mu \pi^a \pi^b \partial^\mu \pi^b}{v^2 - \pi^a \pi^a} \tag{5}$$

Expand this Lagrangian and show that it can be written as

$$\mathcal{L} = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{6v^2}\left((\pi^a \partial_\mu \pi^a)^2 - \pi^a \pi^a \partial_\mu \pi^b \partial^\mu \pi^b\right) + \mathcal{O}(\pi^6) \tag{6}$$

f) In the so-called exponential representation with an SU(2) field U

$$U = \exp\left\{ i\frac{\pi^a \sigma^a}{f_\pi} \right\} , \tag{7}$$

with $f_\pi \approx v$, we can construct the effective Lagrangian for the non-linear sigma model

$$\mathcal{L} = \frac{v^2}{4}\mathrm{Tr}\left(\partial_\mu U^\dagger \partial^\mu U\right) , \tag{8}$$

which is also called chiral Lagrangian. Convince yourself that this object is invariant under SU(2). Under what representation of SU(2) do the fields π^a transform ?

g) Argue that we can organise the chiral Lagrangian in terms of the number of derivatives of the exponential $U(x)$. Why there are no terms containing only $U(x)$ and without derivatives ?

h) Show that a pion mass term can be introduced in this language by adding

$$\delta\mathcal{L}_{\text{mass}} = v^3 \mathrm{Tr}\left(MU + M^\dagger U^\dagger\right) \tag{9}$$

where

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} e^{-i\theta/2} . \tag{10}$$

Argue that we cannot remove the phase by making an $SU(2)_L \times SU(2)_R$ transformation. However, we can trust the experimental value $|\theta| < 10^{-9}$ and set it to zero. Guess how M should transform in such a way that $\delta\mathcal{L}_{\text{mass}}$ is invariant under $SU(2)_L \times SU(2)_R$.