**Problem 1:** fundamental rep. of $SO(3)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i - V(\phi)$$

$$V(\phi) = -\frac{\mu^2}{2}\phi^i\phi^i + \frac{\lambda}{4}(\phi^i\phi^i)^2 \quad , \quad \mu^2, \lambda > 0$$

**(1)**

- Lets write $\Phi = \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix}$

$$\rightarrow \mathcal{L} = \frac{1}{2}\partial_\mu \Phi^T \partial^\mu \Phi + \frac{\mu^2}{2}\Phi^T\Phi - \frac{\lambda}{4}(\Phi^T\Phi)^2$$

- $SO(3)$ rotations act on $\Phi$ as

$$\Phi \longmapsto O\Phi \qquad (\Phi^T \longmapsto (O\Phi)^T = \Phi^T O^T)$$

with $O^T O = \mathbb{1}$

$\partial_\mu O = 0$, because $SO(3)$ global

- $$\mathcal{L} \longmapsto \frac{1}{2}(\partial_\mu \Phi^T)\underbrace{O^T O}_{=\mathbb{1}}(\partial^\mu \Phi) + \frac{\mu^2}{2}\Phi^T\underbrace{O^T O}_{=\mathbb{1}}\Phi - \frac{\lambda}{4}(\Phi^T\underbrace{O^T O}_{=\mathbb{1}}\Phi)^2$$

$$= \mathcal{L}$$

$\rightarrow$ Lagrangian is invariant

**(2)**

- $$\frac{\partial V}{\partial \phi^i} = -\mu^2\phi^i + \lambda(\phi^j\phi^j)\phi^i \overset{!}{=} 0$$

$$\rightarrow \phi^i = 0 \quad \text{or} \quad \phi^j\phi^j = \frac{\mu^2}{\lambda}$$

- $$\frac{\partial V}{\partial \phi^j \partial \phi^i} = -\mu^2\delta_{ij} + 2\lambda\phi^j\phi^i + \lambda\delta_{ij}(\phi^k\phi^k)$$

$$\left.\frac{\partial V}{\partial\phi^j\partial\phi^i}\right|_{\phi^i=0} \leq 0 \quad \rightarrow \quad \phi^i=0 \text{ maximum}$$

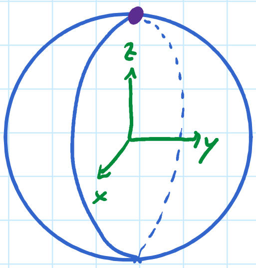
$$\left.\frac{\partial V}{\partial\phi^j\partial\phi^i}\right|_{\phi^i\phi^i=\frac{\mu^2}{\lambda}} \geq 0 \quad \rightarrow \quad \phi^i\phi^i=\frac{\mu^2}{\lambda} \text{ minima}$$

$\rightarrow$ ground state has to satisfy $\phi^i\phi^i=\frac{\mu^2}{\lambda}$

- vacuum manifold: $\mathcal{M}=\{\phi:\ \phi^i\phi^i=\frac{\mu^2}{\lambda}\}\cong S^2$

(3)

- The vacuum manifold is a 2-sphere:




Let us choose the vacuum to be at the purple point. Then this point stays there for rotations around the z-axis. But rotations around the x- and y-axes would rotate the point away. So there is only one rotational symmetry $\rightarrow$ $SO(3)$ is broken to $SO(2)$.

- We broke two symmetries and so there should be two massless Goldstone bosons.

- Lets choose the vev $\Phi_0=(0,0,v)^T$

and expand around it:

$$\Phi = \begin{pmatrix} G_1 \\ G_2 \\ v+h \end{pmatrix} \qquad \text{where} \qquad v^2 = \frac{\mu^2}{\lambda}$$

Inserting into Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu G_1)^2 + \frac{1}{2}(\partial_\mu G_2)^2 + \frac{1}{2}(\partial_\mu (v+h))^2$$
$$+ \frac{\mu^2}{2}\left(G_1^2 + G_2^2 + (v+h)^2\right) - \frac{\lambda}{4}\left(G_1^2 + G_2^2 + \underbrace{(v+h)^2}_{v^2 + 2vh + h^2}\right)^2$$

- Keep only quadratic order:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu G_1)^2 + \frac{1}{2}(\partial_\mu G_2)^2 + \frac{1}{2}(\partial_\mu h)^2$$
$$+ \cancel{\frac{\mu^2}{2}(G_1^2 + G_2^2)} + \frac{\mu^2}{2} h^2$$
$$- \cancel{\frac{\lambda}{2} v^2 (G_1^2 + G_2^2)} - \lambda v^2 h^2 + \dots$$

$$\mu^2 = \lambda v^2$$

$$= \frac{1}{2}(\partial_\mu G_1)^2 + \frac{1}{2}(\partial_\mu G_2)^2 + \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}\lambda v^2 h^2$$

→ two massless Goldstone boson and one massive Higgs boson with mass $m_h^2 = \lambda v^2$

(4)

- Under a local SO(3) transformation we get

$$\partial_\mu \Phi \longmapsto \partial_\mu (O\Phi) = O\,\partial_\mu \Phi + (\partial_\mu O)\Phi$$

The covariant derivative should transform as

$$D_\mu \Phi \longmapsto O\, D_\mu \Phi$$

Thus, we need a term that cancels $(\partial_\mu O)\Phi$.

$$D_\mu \Phi = \partial_\mu \Phi + g A_\mu \Phi$$
$$\longmapsto O\,\partial_\mu \Phi + (\partial_\mu O)\Phi + g A'_\mu O \Phi$$

$$\longmapsto O\,\partial_\mu\Phi + (\partial_\mu O)\Phi + g A'_\mu O\Phi$$

$$\stackrel{!}{=} O\partial_\mu\Phi + g\,O A_\mu \Phi$$

$$\rightarrow A'_\mu = O A_\mu O^T - \frac{1}{g}(\partial_\mu O) O^T$$

- The field strength tensor is defined by

$$F_{\mu\nu}\Phi = \frac{1}{g}[D_\mu, D_\nu]\Phi$$

$$= \frac{1}{g}[\partial_\mu + gA_\mu\, ,\, \partial_\nu + gA_\nu\,]\Phi$$

$$= \frac{1}{g}\,\cancel{[\partial_\mu,\partial_\nu]}\Phi + [A_\mu, \partial_\nu]\Phi + [\partial_\mu, A_\nu]\Phi + g[A_\mu, A_\nu]\Phi$$

$\partial\Phi$ terms cancel

$$= (\partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu])\Phi$$

- The Lagrangian that is invariant under local $SU(3)$ symmetry is

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2}(D_\mu\Phi)^T(D^\mu\Phi) - V(\Phi)$$

with

$$D_\mu\Phi = \partial_\mu\Phi + gA_\mu\Phi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]$$

(5)

- $\frac{1}{2}\,(D_\mu(\Phi+\Phi_0))^T(D^\mu(\Phi+\Phi_0))$

$$\supset \frac{1}{2}(D_\mu\Phi_0)^T(D^\mu\Phi_0)$$

$$= \frac{1}{2}g^2\,\Phi_0^T A_\mu^T A^\mu\,\Phi_0$$

$$= \frac{1}{2}g^2 A_\mu^a A^{b\mu}\quad \Phi_0^T\,T^aT^b\,\Phi_0$$

$$T^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T^1 T^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 T^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^1 T^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^1 T^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 T^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T^3 T^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{L} \supset \frac{1}{2} g^2 v^2 \left( A^2_\mu A^{2\mu} + A^3_\mu A^{3\mu} \right)$$

$$\rightarrow \quad m_V = gv$$

- the mass matrix has one vanishing eigenvalue that corresponds to one massless gauge boson. The massive gauge bosons have the mass $m_V = gv$.

## Problem 2:

real scalar field

- $$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi_1 \partial^\mu \Phi_1 + \frac{1}{2} \partial_\mu \Phi_2 \partial^\mu \Phi_2 + \frac{\mu^2}{2} \left( \Phi_1^2 + \Phi_2^2 \right) - \frac{\lambda}{4} \left( \Phi_1^2 + \Phi_2^2 \right)^2 + \varepsilon U(\Phi_1)$$

(1)

- $\varepsilon = 0$:

Let us define the field $\Psi := \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$.

We can write the Lagrangian as follows

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Psi^T \partial^\mu \Psi + \frac{\mu^2}{2} \Psi^T \Psi - \frac{\lambda}{4} \left( \Psi^T \Psi \right)^2$$

$\rightarrow$ The symmetry of the theory is $SO(2)$ with $\Psi \longmapsto O\Psi$, where $O$ can be with $\Psi \longmapsto O\Psi$, where $O$ can be written as

$$O = \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix}$$

- $\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \longmapsto \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$

$$= \begin{pmatrix} \cos\Theta\,\Phi_1 - \sin\Theta\,\Phi_2 \\ \sin\Theta\,\Phi_1 + \cos\Theta\,\Phi_2 \end{pmatrix} \approx \begin{pmatrix} \Phi_1 - \Theta\Phi_2 \\ \Phi_2 + \Theta\Phi_1 \end{pmatrix}$$

$\rightarrow \delta\Phi_1 = -\Theta\Phi_2 \quad , \quad \delta\Phi_2 = \Theta\Phi_1$

- Noether current:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_1)}\delta\Phi_1 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_2)}\delta\Phi_2$$

$$\sim -(\partial^\mu \Phi_1)\Phi_2 + (\partial^\mu \Phi_2)\Phi_1$$

- $V = -\frac{\mu^2}{2}(\Phi_1^2 + \Phi_2^2) + \frac{\lambda}{4}(\Phi_1^2 + \Phi_2^2)^2$

$$= -\frac{\mu^2}{2}\Phi_i\Phi_i + \frac{\lambda}{4}(\Phi_i\Phi_i)^2$$

$$\frac{\partial V}{\partial \Phi_i} = -\mu^2 \Phi_i + \lambda(\Phi_j\Phi_j)\Phi_i \overset{!}{=} 0$$

$\rightarrow \Phi_i = 0 \quad$ or $\quad \Phi_i\Phi_i = \frac{\mu^2}{\lambda}$

$$\frac{\partial^2 V}{\partial\Phi_j \partial\Phi_i} = -\mu^2\delta_{ij} + \lambda(\Phi_j\Phi_j)\delta_{ij} + 2\lambda\Phi_i\Phi_j$$

$$\left.\frac{\partial^2 V}{\partial\Phi_j \partial\Phi_i}\right|_{\Phi_i = 0} = -\mu^2\delta_{ij} \rightarrow \text{eigenvalues} \leq 0 \rightarrow \text{maximum}$$

$$\left.\frac{\partial V}{\partial \Phi_j \partial \Phi_i}\right|_{\Phi_1 = \frac{\mu}{\sqrt{\lambda}}, \Phi_2 = 0} = -\mu^2 \delta_{ij} + \mu^2 \delta_{ij} + 2\mu^2 \delta_{i1}\delta_{j1} \geq 0 \to \text{minimum}$$

choice on circle $\Phi_i \Phi_i = \frac{\mu^2}{\lambda}$

- The ground states are $\{\Phi_1, \Phi_2 : \Phi_1^2 + \Phi_2^2 = \frac{\mu^2}{\lambda}\}$.

If we choose $\langle \Phi_1 \rangle = \frac{\mu}{\sqrt{\lambda}}$, $\langle \Phi_2 \rangle = 0$, then the perturbation $\chi$ around $\langle \Phi_2 \rangle$: $\Phi_2 = \langle \Phi_2 \rangle + \chi$ is the massless Nambu-Goldstone boson.

(2)

- $\varepsilon \neq 0$

$$\to V = -\frac{\mu^2}{2}(\Phi_1^2 + \Phi_2^2) + \frac{\lambda}{4}(\Phi_1^2 + \Phi_2^2)^2 - \varepsilon U(\Phi_1)$$

- $$\frac{\partial V}{\partial \Phi_2} = -\mu^2 \Phi_2 + \lambda(\Phi_1^2 + \Phi_2^2)\Phi_2 \overset{!}{=} 0$$

$$\frac{\partial V}{\partial \Phi_1} = -\mu^2 \Phi_1 + \lambda(\Phi_1^2 + \Phi_2^2)\Phi_1 - \varepsilon U'(\Phi_1) \overset{!}{=} 0$$

- There are two possibilities to minimize the potential:

(i) $\Phi_2 = 0$

$$-\mu^2 \Phi_1 + \lambda \Phi_1^3 - \varepsilon U'(\Phi_1) = 0$$

$$\Phi_1 \approx a + \varepsilon b$$

$$\to -\mu^2 a - \mu^2 \varepsilon b + \lambda a^3 + 3\lambda a^2 b \varepsilon - \varepsilon U' \approx 0$$

$$\to a = \frac{\mu}{\sqrt{\lambda}}, \quad b = +\frac{U'}{2\mu^2}$$

$$\to \Phi_1 = \frac{\mu}{\sqrt{\lambda}} + \varepsilon \frac{U'}{2\mu^2}$$

$$\to \Phi_1 = \frac{\mu}{\sqrt{\lambda}} + \varepsilon \frac{U'}{2\mu^2}$$

(ii) $\Phi_2 = \frac{\mu}{\sqrt{\lambda}}, \quad \Phi_1 = 0, \quad U' = 0$

- In both cases we obtain a light boson with mass proportional to $\varepsilon$ $\to$ pseudo Goldstone boson

$$\frac{\partial^2 V}{\partial\Phi_i \partial\Phi_j} = \begin{pmatrix} -\mu^2 + \lambda(\Phi_1^2 + \Phi_2^2) + 2\lambda\Phi_1^2 + \varepsilon U'' & 2\lambda\Phi_1\Phi_2 \\ 2\lambda\Phi_1\Phi_2 & -\mu^2 + \lambda(\Phi_1^2 + \Phi_2^2)^2 + 2\lambda\Phi_2^2 \end{pmatrix}$$

(i) $$\frac{\partial^2 V}{\partial\Phi_i \partial\Phi_j} = \begin{pmatrix} 2\mu^2 + 3\varepsilon\frac{\sqrt{\lambda}}{\mu}U' + \varepsilon U'' & 0 \\ 0 & \varepsilon\frac{\sqrt{\lambda}}{\mu}U' \end{pmatrix}$$

(ii) $$\frac{\partial^2 V}{\partial\Phi_i \partial\Phi_j} = \begin{pmatrix} \varepsilon U'' & 0 \\ 0 & 2\mu^2 \end{pmatrix}$$