Standard Model and QCD

Problem Sheet 1

23 April 2024

1. The $SU(N)$ group

Consider the elements of $SU(N)$, characterized by $N \times N$ matrices $U = e^{i\lambda^a T^a}$ satisfying $U^\dagger U = 1$, $\det U = 1$. Here λ^a are real parameters and the generators T^a are $(N \times N)$ matrices with complex entries $(a = 1, ..., k)$.

1. Discuss the constraints that the previous defining conditions impose on the matrices T^a and identify the number of generators k for $SU(N)$ groups.
2. Use the previous results to construct explicitly the generators T^a of $SU(2)$ and $SU(3)$. Normalize them so that $\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}$
3. What is the rank of a group? Specify the rank of $SU(N)$ and identify the Casimir operators for $SU(2)$ and $SU(3)$.
4. Define the fundamental and the adjoint representations of $SU(N)$ according to their transformation rules. Specify their dimension.
5. The generators T^a span the space of group transformations which are infinitesimally close to the identity. The commutation relations between the generators can be written as $[T^a, T^b] = i f^{abc} T^c$, and define the algebra of the group. Here, the numbers f^{abc} are called *structure constants*. Check for the cases $SU(2)$ and $SU(3)$ that the algebra closes and compute the structure constants for these cases. Are the structure constants related to the adjoint representation?

2. Gauge Theories

Let us consider the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi \ . \tag{1}$$

1. Check that the above is invariant under the global $U(1)$ transformations
$$\psi(x) \longrightarrow \psi'(x) = e^{-ie\alpha}\psi(x). \tag{2}$$
with e the charge of the field and α a constant. Find the corresponding Noether current j_μ and check that it is conserved on the equations of motion. Compute the associated charge Q.
2. Verify that the Dirac Lagrangian is not invariant under the local $U(1)$ transformation
$$\psi(x) \longrightarrow \psi'(x) = e^{-ie\alpha(x)}\psi(x). \tag{3}$$
3. Show that (1) becomes invariant under the local $U(1)$ once we supplement it with a term proportional to $j^\mu A_\mu$, with j_μ the Noether current that you computed previously and A_μ the $U(1)$ gauge field transforming as
$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\alpha(x) \ . \tag{4}$$

4. Show that adding $j^\mu A_\mu$ is equivalent to replacing the partial derivative with a covariant one.
5. Write down the Lagrangian for QED and find the equations of motion for the fields.
6. Using the equations of motion show that the Noether charge (*which is associated with the global $U(1)$ symmetry!*) can be written as a surface integral at spatial infinity.