<u>MPS.9-12: Translationally invariant MPS</u> [Schollwöck2011, Sec. 4.2.2] MPS.9

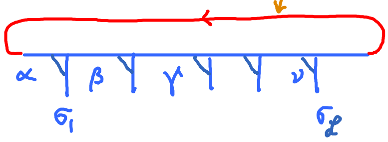
Consider translationally invariant MPS, e.g. infinite system, or length-$\mathcal{L}$ chain with periodic boundary conditions. Then all tensors defining the MPS are identical: $A_{[\ell]} = A$ for all ℓ.

Goal: compute matrix elements and correlation functions for such a system.

MPS.9 Transfer matrix

Consider length-$\mathcal{L}$ chain with <u>periodic</u> boundary conditions (and A's not necessarily all equal):

indicates trace



$$|\psi\rangle = |\vec{\sigma}_{\mathcal{L}}\rangle [A_1]^{\alpha\sigma_1}{}_{\beta}[A_2]^{\beta\sigma_2}{}_{\gamma} \dots [A_{\mathcal{L}}]^{\nu\sigma_{\mathcal{L}}}{}_{\alpha}$$

$$\equiv |\vec{\sigma}_{\mathcal{L}}\rangle \,\mathrm{Tr}\left[A_1^{\sigma_1} A_2^{\sigma_2} \dots A_{\mathcal{L}}^{\sigma_{\mathcal{L}}}\right]$$

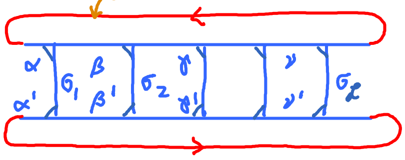
All bonds have same dimension: $D_\alpha = D_\beta = D_\gamma = D_\nu =: D$ (1)

This is assumed throughout below.

Normalization:

indicates trace

$$\langle\psi|\psi\rangle = \quad (2)$$



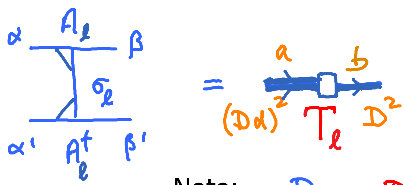
$$\overline{[A_1]^{\alpha'\sigma_1}{}_{\beta'}}\,\overline{[A_2]^{\beta'\sigma_2}{}_{\gamma'}}\dots\overline{[A_{\mathcal{L}}]^{\nu'\sigma_{\mathcal{L}}}{}_{\alpha'}}\;[A_1]^{\alpha\sigma_1}{}_{\beta}[A_2]^{\beta\sigma_2}{}_{\gamma}\dots[A_{\mathcal{L}}]^{\nu\sigma_{\mathcal{L}}}{}_{\alpha} \quad (3)$$

regroup

$$= \underbrace{\left([A_1^\dagger]^{\beta'}{}_{\sigma_1\alpha'}[A_1]^{\alpha\sigma_1}{}_{\beta}\right)}_{:=[T_1]_{\alpha'}{}^{\alpha\beta'}{}_{\beta}}\underbrace{\left([A_2]^{\dagger\gamma'}{}_{\sigma_2\beta'}[A_2]^{\beta\sigma_2}{}_{\gamma}\right)}_{:=[T_2]_{\beta'}{}^{\beta\gamma'}{}_{\gamma}}\dots\underbrace{\left([A_{\mathcal{L}}^\dagger]^{\alpha'}{}_{\sigma_{\mathcal{L}}\nu'}[A_{\mathcal{L}}]^{\nu\sigma_{\mathcal{L}}}{}_{\alpha}\right)}_{:=[T_{\mathcal{L}}]_{\gamma'}{}^{\nu\alpha'}{}_{\alpha}} \quad (4)$$

We defined the 'transfer matrix' (with collective indices chosen to reflect arrows on effective vertex)

$$[T_\ell]^a{}_b := [T_\ell]\underbrace{{}_{\alpha'}{}^{\alpha}}_{a}\underbrace{{}^{\beta'}{}_{\beta}}_{b} := \underbrace{[A_\ell]^{\dagger\beta'}{}_{\sigma_\ell\alpha'}}_{\overset{\shortparallel}{\overline{[A_\ell]^{\alpha'\sigma_\ell}{}_{\beta'}}}}[A_\ell]^{\alpha\sigma_\ell}{}_{\beta} \quad (5)$$



Note: $D_\mu = D^2$ (6)

Then

$$\langle\psi|\psi\rangle = [T_1]^a{}_b[T_2]^b{}_c\dots[T_{\mathcal{L}}]^n{}_a = \mathrm{Tr}\left(T_1 T_2 \dots T_{\mathcal{L}}\right) \quad (7)$$

Assume all A-tensors are identical, then the same is true for all T-matrices. Hence

$$\langle\psi|\psi\rangle = \mathrm{Tr}\left(T^{\mathcal{L}}\right) = \sum_j (t_j)^{\mathcal{L}} \xrightarrow{\mathcal{L}\to\infty} (t_1)^{\mathcal{L}} \quad (8)$$

where t_j are the left eigenvalues of the transfer matrix, and t_1 is the largest one of these.

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Assume now that A -tensor is left-normalized (analogous discussion holds if it is right-normalized).

Then we know that the MPS is normalized to unity: $$1 \overset{\text{(MPS.4.3b)}}{=} \langle \psi | \psi \rangle \qquad (9)$$

(MPS.9.8) implies for largest eigenvalue of transfer matrix: $$(t_1)^{\mathcal{L}} = 1 \quad \Rightarrow \quad t_1 = 1 . \qquad (10)$$

Hence, all eigenvalues of transfer matrix satisfy $$\boxed{|t_j| \leq 1} . \qquad (11)$$

eigenvector label: j = 1
components of eigenvector

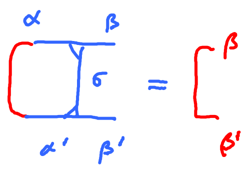
Claim: the left eigenvector with eigenvalue $t_{j=1} = 1$, say $V^{j=1}$, is $$[V^1]_\alpha{}^{\alpha'} := \mathbb{1}^{\alpha'}{}_{\alpha} \qquad (12)$$

Check: do we find $V_a T^a{}_b = V_b$?

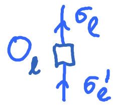
'vector in transfer space' = 'matrix in original space'

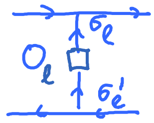
$$[V^1]_a T^a{}_b = A^{\dagger \beta'}{}_{\sigma\alpha'} \, \mathbb{1}^{\alpha'}{}_{\alpha} \, A^{\alpha\sigma}{}_{\beta} \qquad (13)$$

$$= A^{\dagger \beta'}{}_{\sigma\alpha} \, A^{\alpha\sigma}{}_{\beta} = \mathbb{1}^{\beta'}{}_{\beta} = [V^1]_b \; \checkmark \qquad (14)$$



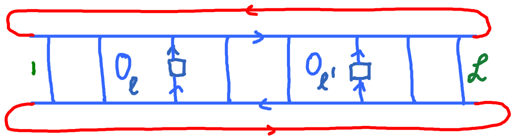
Correlation functions

Consider local operator: $\hat{O}_\ell = |\sigma'_\ell\rangle [O_\ell]^{\sigma'_\ell}{}_{\sigma_\ell} \langle \sigma_\ell|$  (5)

Define corresponding transfer matrix: $T_{O_\ell} = A^\dagger_{\sigma'_\ell} [O_\ell]^{\sigma'_\ell}{}_{\sigma_\ell} A^{\sigma_\ell}$  (6)

Correlator:

$$C_{\ell\ell'} := \langle\psi| \hat{O}_\ell \hat{O}_{\ell'} |\psi\rangle = \qquad (7)$$



$$= \mathrm{Tr}\left(T^{\ell-1} T_{O_\ell} T^{\ell'-\ell-1} T_{O_{\ell'}} T^{\mathcal{L}-\ell'}\right) \underset{\text{cyclic invariance of trace}}{=} \mathrm{Tr}\left(T^{\mathcal{L}-(\ell'-\ell)-1} T_{O_\ell} T^{\ell'-\ell-1} T_{O_{\ell'}}\right) \qquad (8)$$

Let V^j, t_j be left eigenvectors, eigenvalues of transfer matrix: $V^j T = t_j V^j$ (9)

[or explicitly, with matrix indices: $[V^j]_a T^a{}_b = t_j [V^j]_b$] <u>no</u> sum on j here (10)

Transform to eigenbasis of transfer matrix:

$$C_{\ell\ell'} = \sum_{jj'} (t_{j'})^{\mathcal{L}-(\ell'-\ell)-1} [T_{O_\ell}]^{j'}{}_j \, (t_j)^{\ell'-\ell-1} [T_{O_{\ell'}}]^j{}_{j'} \qquad (11)$$

For $\mathcal{L} \to \infty$, only contribution of largest eigenvalue, $t_{j'} = t_1 = 1$ (assume $\langle\psi|\psi\rangle = 1$), survives from sum over j' :

$$C_{\ell\ell'} \xrightarrow{\mathcal{L}\to\infty} \sum_j [T_{O_\ell}]^1{}_j \, t_j^{\ell'-\ell-1} [T_{O_{\ell'}}]^j{}_1 \qquad (12)$$

Assume $\hat{O}_{\ell'} = \hat{O}^\dagger_\ell \equiv \hat{O}$, and take their separation to be large, $\ell' - \ell \to \infty$ (13)

$$C_{\ell\ell'} \xrightarrow{\ell'-\ell\to\infty} |[T_O]^1{}_1|^2 + |[T_O]^1{}_2|^2 (t_2)^{\ell'-\ell-1} + \dots \qquad (14)$$

If $[T_O]^1{}_1 \neq 0$: 'long-range order' (15)

If $[T_O]^1{}_1 = 0$: 'exponential decay', $\sim e^{-|\ell'-\ell|/\xi}$ (16)

with correlation length $\xi = [\ln(t_1/t_2)]^{-1}$ (17)

MPS.10 AKLT Model

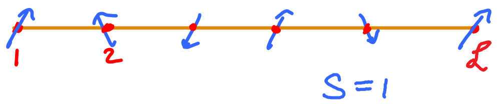
[Affleck1988], [Schollwöck2011, Sec. 4.1.5], [Tu2008]

(thanks to Hong-Hao Tu for notes!)

General remarks

- AKLT model was proposed by Affleck, Kennedy, Lieb, Tasaki in 1988.
- Previously, Haldane had predicted that S=1 Heisenberg spin chain has finite excitation gap above a unique ground state, i.e. only 'massive' excitations [Haldane1983a], [Haldane1983b].
- AKLT then constructed the first solvable, isotropic, S=1 spin chain model that exhibits a 'Haldane gap'.
- Ground state of AKLT model is an MPS of lowest non-trivial bond dimension, D=2.
- Correlation functions decay exponentially - the correlation length can be computed analytically.

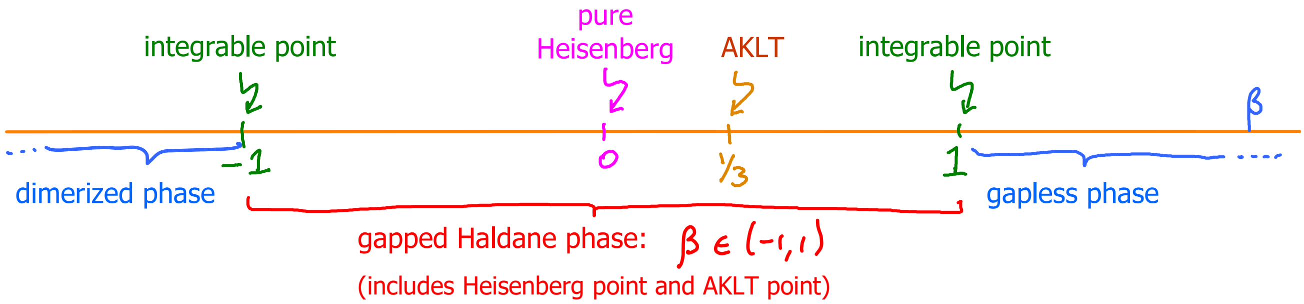
Haldane phase for S=1 spin chains



Consider bilinear-biquadratic (BB) Heisenberg model for 1D chain of spin S=1:

$$H_{BB} = \sum_{\ell=1}^{\mathcal{L}-1} \vec{S}_\ell \cdot \vec{S}_{\ell+1} + \beta \left(\vec{S}_\ell \cdot \vec{S}_{\ell+1} \right)^2 \qquad (1)$$

Phase diagram:



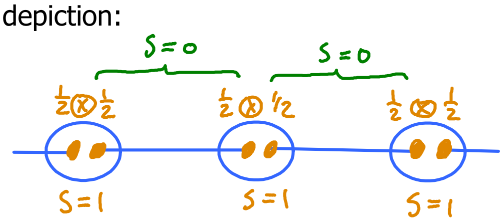
Main idea of AKLT model: $H_{AKLT} = H_{BB}\,(\beta = 1/3)$ (2)

is built from projectors mapping spins on neighboring sites to total spin $S^{tot}_{\ell\,\ell+1} = 2$.

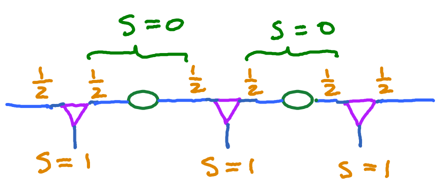
Ground state satsifies $H_{AKLT} |g\rangle = 0$. To achieve this, ground state is constructed in such a manner that spins on neighboring sites can only be coupled to $S^{tot}_{\ell,\ell+1} = 0$ or 1.

To this end, the spin-1 on each site is constructed from two auxiliary spin-1/2 degrees of freedom; One spin-1/2 each from neighboring sites is coupled to spin 0; this projects out the S=2 sector in the direct-product space of neighboring sites, ensuring that H_{AKLT} annihilates ground state.

traditional depiction:



MPS depiction: spin-1/2's live on bonds



(3)

Construction of AKLT Hamiltonian

Direct product space of spin 1 with spin 1 contains direct sum of spin 0, 1, 2:

$$\mathbb{V}_1 \otimes \mathbb{V}_1 = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2 \qquad (4)$$

here, subscripts label spin representations (site indices are suppressed)

Projector of $\mathbb{V}_1 \otimes \mathbb{V}_1$ onto $\mathbb{V}_S$ (with $S = 0, 1, 2$):

$$\underbrace{P^{(S)}_{1,2}}_{\text{sites 1,2}} = P^{(S)}_{1,2}(\vec{S}_1 \cdot \vec{S}_2) := \underbrace{c}_{\text{normalization factor}} \prod_{S' \neq S} \Big[(\vec{S}_1 + \vec{S}_2)^2 \underbrace{-\, S'(S'+1)}_{\text{yields zero when total spin} = S'} \Big] \qquad (5)$$

Using $(\vec{S}_1 + \vec{S}_2)^2 = \underbrace{\vec{S}_1^{\,2}}_{1(1+1)} + 2\,\vec{S}_1 \cdot \vec{S}_2 + \underbrace{\vec{S}_2^{\,2}}_{1(1+1)} = 2\,\vec{S}_1 \cdot \vec{S}_2 + 4$, we find for spin-2 projector: (6)

$$P^{(2)}_{1,2} \overset{(5)}{=} c\,\big[\,2\,\vec{S}_1 \cdot \vec{S}_2 + 4 - 0(0+1)\big]\big[\,2\,\vec{S}_1 \cdot \vec{S}_2 + \underbrace{4 - 1(1+1)}_{2}\big] \qquad (7)$$

$$= c\,\big\{\,4\,(\vec{S}_1 \cdot \vec{S}_2)^2 + 12\,\vec{S}_1 \cdot \vec{S}_2 + 8\,\big\} \qquad (8)$$

Normalization is fixed by demanding that $P^{(2)}_{1,2}$ must yield 1 when acting on spin-2 subspace:

$$1 = P^{(2)}_{1,2}\Big|_{(\vec{S}_1 + \vec{S}_2)^2 = 2(2+1)} \overset{(5)}{=} c\,\big[\,2(2+1) - 0\,\big]\big[\,2(2+1) - 1(1+1)\,\big] \qquad (9)$$

$$\Rightarrow \qquad = c \; 6 \cdot 4 \qquad \Rightarrow \quad c = \frac{1}{24} \qquad (10)$$

$$\boxed{P^{(2)}_{1,2} \overset{(8)}{=} \frac{1}{6}(\vec{S}_1 \cdot \vec{S}_2)^2 + \frac{1}{2}\,\vec{S}_1 \cdot \vec{S}_2 + \frac{1}{3} =: P^{(2)}_{1,2}(\vec{S}_1, \vec{S}_2) = \text{projector on spin-2 subspace}} \qquad (11)$$

AKLT Hamiltonian is sum over spin-2 projectors for all neighboring pairs of spins.

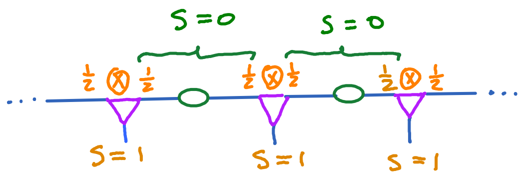
$$H_{AKLT} = \sum_{\ell} P^{(2)}_{\ell,\ell+1}(\vec{S}_\ell \cdot \vec{S}_{\ell+1}) \qquad (12)$$

For a finite chain of $\mathcal{L}$ sites, use periodic boundary conditions, i.e. identify $\vec{S}_{\ell+\mathcal{L}} = \vec{S}_\ell$.

Each term is a projector, hence has only non-negative eigenvalues. Hence same is true for H_{AKLT} .

$\Rightarrow$ A state satisfying $H_{AKLT}|\psi\rangle = 0\,|\psi\rangle = 0$ must be a ground state!

MPS.11 AKLT ground state



(1)

On every site, represent spin 1 as symmetric combination of two auxiliary spin-1/2 degrees of freedom:

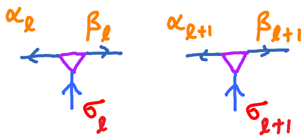
$$|S=1,\sigma\rangle \equiv |\sigma\rangle = \begin{cases} |+1\rangle = |\uparrow\rangle|\uparrow\rangle \\ |0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\right) \\ |-1\rangle = |\downarrow\rangle|\downarrow\rangle \end{cases} \qquad (2)$$

On-site projector that maps $\mathbb{V}_{\frac{1}{2}} \otimes \mathbb{V}_{\frac{1}{2}}$ to $\mathbb{V}_1$:

$$\hat{C} = |+1\rangle\langle\uparrow|\langle\uparrow| + |0\rangle\frac{1}{\sqrt{2}}\left(\langle\uparrow|\langle\downarrow| + \langle\downarrow|\langle\uparrow|\right) + |-1\rangle\langle\downarrow|\langle\downarrow| \qquad (3)$$

Use such a projector on every site ℓ :

$$\hat{C}_\ell = |\sigma_\ell\rangle_\ell \, [C]^{\sigma_\ell}{}_{\alpha_\ell \beta_\ell} \; {}_\ell\langle\alpha_\ell| \, {}_\ell\langle\beta_\ell| \qquad (4)$$

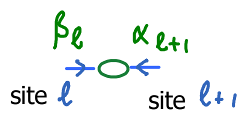


with $C^{+1} = \begin{array}{c|cc} & \uparrow & \downarrow \\ \hline \uparrow & 1 & 0 \\ \downarrow & 0 & 0 \end{array}$ ($\alpha = \beta = \uparrow$), $\quad C^{0} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ($\alpha \neq \beta$), $\quad C^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ($\alpha = \beta = \downarrow$) $\longleftarrow$ Clebsch-Gordan Coefficients for coupling $\frac{1}{2} \otimes \frac{1}{2} \to 1$ (5)

Haldane: 'neighbors shake hands'

Now construct nearest-neighbor 'valence bonds' built from auxiliary spin-1/2 states:

$$|V\rangle_\ell = |\beta_\ell\rangle_\ell \, |\alpha_{\ell+1}\rangle_{\ell+1} \, V^{\beta_\ell \alpha_{\ell+1}} \equiv \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_\ell |\downarrow\rangle_{\ell+1} - |\downarrow\rangle_\ell |\uparrow\rangle_{\ell+1}\right) \qquad (6)$$

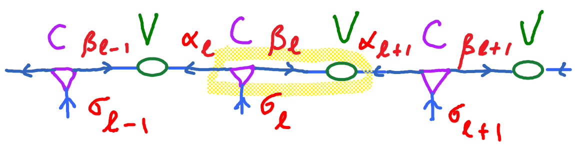


$$V = \frac{1}{\sqrt{2}} \begin{array}{c|cc} & \uparrow & \downarrow \\ \hline \uparrow & 0 & 1 \\ \downarrow & -1 & 0 \end{array}$$

Haldane: 'each site hand-shakes with its neighbors'

AKLT ground state = (direct product of spin-1 projectors) acting on (direct product of valence bonds):

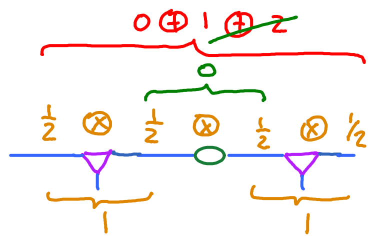
$$|g\rangle \equiv \prod_{\otimes \ell} \hat{C}_\ell \; \prod_{\otimes \ell} |V\rangle_\ell = \cdots$$



(7)

Why is this a ground state?

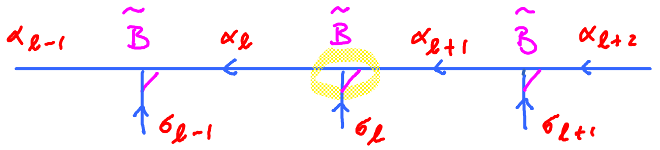
Coupling two auxiliary spin-1/2 to total spin 0 (valence bond) eliminates the spin-2 sector in direct product space of two spin-1, hence spin-2 projector in H_{AKLT} yields zero when acting on this. (Will be checked explicitly below.)



(8)

AKLT ground state is an MPS!

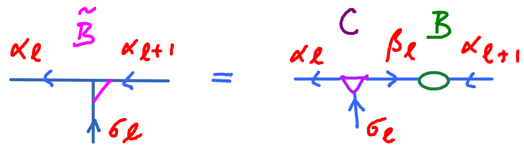
$$\boxed{|\psi\rangle = \prod_{\otimes \ell} |\sigma_\ell\rangle\, \tilde{B}_{\alpha_\ell}{}^{\sigma_\ell \alpha_{\ell+1}}}$$



(9)

with

$$\tilde{B}_{\alpha_\ell}{}^{\sigma_\ell \alpha_{\ell+1}} = C^{\sigma_\ell}{}_{\alpha_\ell \beta_\ell}\, V^{\beta_\ell \alpha_{\ell+1}}$$



(10)

Explicitly: $\sigma_\ell = +1$: $\tilde{B}^{+1} = \begin{pmatrix}1&0\\0&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\0&0\end{pmatrix}$ (11)

$\sigma_\ell = 0$: $\tilde{B}^{0} = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ (12)

$\sigma_\ell = +1$: $\tilde{B}^{-1} = \begin{pmatrix}0&0\\0&1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0\\-1&0\end{pmatrix}$ (13)

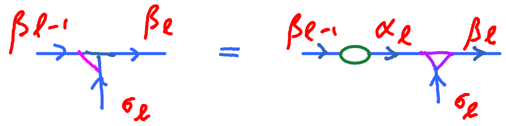
Not normalized: $\tilde{B}_\sigma \tilde{B}^{\dagger\sigma} = \frac{1}{2}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} + \frac{1}{4}\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}-1&0\\0&1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0&0\\-1&0\end{pmatrix}\begin{pmatrix}0&-1\\0&0\end{pmatrix} = \frac{3}{4}\mathbb{1}$ (14)

Define right-normalized tensors, satisfying $B_\sigma B^{\dagger\sigma} = \mathbb{1}$: $B^\sigma := \sqrt{\frac{4}{3}}\,\tilde{B}^\sigma$ (15)

$$\boxed{B^{+1} = \sqrt{\tfrac{2}{3}}\begin{pmatrix}0&1\\0&0\end{pmatrix}, \quad B^{0} = \tfrac{1}{\sqrt{3}}\begin{pmatrix}-1&0\\0&1\end{pmatrix}, \quad B^{-1} = \sqrt{\tfrac{2}{3}}\begin{pmatrix}0&0\\-1&0\end{pmatrix}}$$ (16)

Remark: we could also have grouped B and C in opposite order, defining

$$\tilde{A}^{\beta_{\ell-1}\sigma_\ell}{}_{\beta_\ell} = B^{\beta_{\ell-1}\alpha_\ell}\, C^{\sigma_\ell}{}_{\alpha_\ell\beta_\ell}$$

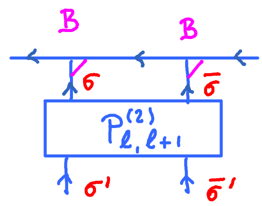


(17)

This leads to left-normalized tensors, with $A^{\pm1} = B^{\mp1}$, $A^0 = B^0$ (18)

<u>Exercise:</u> verify that the projector $P^{(2)}_{\ell,\ell+1}(\vec{S}_\ell, \vec{S}_{\ell+1})$

from (MPS.10.11) yields zero when acting on sites $\ell, \ell+1$ of $|\psi\rangle$

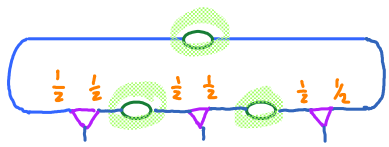


(19)

Hint: use spin-1 representation for $(\vec{S}_\ell\cdot\vec{S}_{\ell+1})^{\sigma\bar{\sigma}}{}_{\sigma'\bar{\sigma}'} = \vec{S}^{\sigma}{}_{\sigma'}\cdot\vec{S}^{\bar{\sigma}}{}_{\bar{\sigma}'}$ (20)

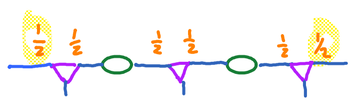
<u>Boundary conditions</u>

For periodic boundary conditions, Hamiltonian includes projector connecting sites 1 and $\mathcal{L}$. Then <u>all</u> auxiliary spin $\frac{1}{2}$ are bound into pairs of singlets, and ground state is unique.



(21)

For <u>open</u> boundary conditions, there are 'left-over spin-1/2' degrees of freedom at both ends of chain. Ground state is four-fold degenerate.

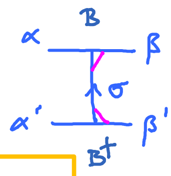


(22)

MPS.12 Transfer operator, string order parameter

(using right-normalized states, in contrast to left-normalized ones from MPS.9)

$$T^{a}{}_{b} = T^{\alpha'}{}_{\alpha\beta'}{}^{\beta} = B^{\dagger}_{\beta'\sigma}{}^{\alpha'} B_{\alpha}{}^{\sigma\beta} = \overline{B_{\alpha'}{}^{\sigma\beta'}}\, B_{\alpha}{}^{\sigma\beta} \qquad (1)$$



$$\boxed{B^{+1} = \sqrt{\tfrac{2}{3}}\begin{pmatrix}0&1\\0&0\end{pmatrix}, \qquad B^{0} = \tfrac{1}{\sqrt{3}}\begin{pmatrix}-1&0\\0&1\end{pmatrix}, \qquad B^{-1} = \sqrt{\tfrac{2}{3}}\begin{pmatrix}0&0\\-1&0\end{pmatrix}} \qquad (2)$$

B^σ is real, hence $\overline{B^\sigma} = B^\sigma$

$$T = \overline{B^\sigma} \otimes B^\sigma = B^{+1}\otimes B^{+1} + B^{0}\otimes B^{0} + B^{-1}\otimes B^{-1} \qquad (3)$$

$$= \sqrt{\tfrac{2}{3}}\left(\begin{array}{c|c} 0 & 1\cdot\sqrt{\tfrac{2}{3}}\begin{pmatrix}0&1\\0&0\end{pmatrix} \\ \hline 0 & 0 \end{array}\right)_{\sigma=1} + \frac{1}{\sqrt{3}}\left(\begin{array}{c|c} -1\,\tfrac{1}{\sqrt{3}}\begin{pmatrix}-1&0\\0&1\end{pmatrix} & 0 \\ \hline 0 & 1\cdot\tfrac{1}{\sqrt{3}}\begin{pmatrix}-1&0\\0&1\end{pmatrix} \end{array}\right)_{\sigma=0} + \sqrt{\tfrac{2}{3}}\left(\begin{array}{c|c} 0 & 0 \\ \hline -1\cdot\sqrt{\tfrac{2}{3}}\begin{pmatrix}0&0\\-1&0\end{pmatrix} & 0 \end{array}\right)_{\sigma=-1} \qquad (4)$$

$$= \frac{1}{3}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right) \qquad (5)$$

To compute spin-spin correlator, $C^{zz}_{\ell\ell'} = \dfrac{\langle g|S^z_\ell\; S^z_{\ell'}\;|g\rangle}{\langle g|g\rangle}$, we need (6)

$$T_{S^z} = B^{\dagger}_{\sigma'}\,(S^z)^{\sigma'}{}_{\sigma}\,B^\sigma \quad , \quad \text{with} \quad S^z = \begin{pmatrix}1&&\\&0&\\&&-1\end{pmatrix} \qquad (7)$$

$$= 1\cdot\sqrt{\tfrac{2}{3}}\left(\begin{array}{c|c} 0 & 1\cdot\sqrt{\tfrac{2}{3}}\begin{pmatrix}0&1\\0&0\end{pmatrix} \\ \hline 0 & 0 \end{array}\right)_{\sigma=\sigma'=1} + 0\cdot\frac{1}{\sqrt{3}}\left(\begin{array}{c|c} -1\,\tfrac{1}{\sqrt{3}}\begin{pmatrix}-1&0\\0&1\end{pmatrix} & 0 \\ \hline 0 & 1\cdot\tfrac{1}{\sqrt{3}}\begin{pmatrix}-1&0\\0&1\end{pmatrix} \end{array}\right)_{\sigma=\sigma'=0} + (-1)\cdot\sqrt{\tfrac{2}{3}}\left(\begin{array}{c|c} 0 & 0 \\ \hline -1\cdot\sqrt{\tfrac{2}{3}}\begin{pmatrix}0&0\\-1&0\end{pmatrix} & 0 \end{array}\right)_{\sigma=\sigma'=-1} \qquad (8)$$

$$= \frac{2}{3}\left(\begin{array}{c|c} 0 & \begin{matrix}0&1\\0&0\end{matrix} \\ \hline \begin{matrix}0&0\\-1&0\end{matrix} & 0 \end{array}\right) \qquad (9)$$

Exercise

(a) Compute the eigenvalues and eigenvectors of T (10)

(b) Show that $C^{zz}_{\ell\ell'} \sim e^{-|\ell'-\ell|/\xi}$, with $\xi = \dfrac{1}{\ln 3}$ (11)

Remark: since the correlation length is finite, the model is gapped!

String order parameter

AKLT ground state: $|g\rangle = |\vec{\sigma}_{\mathscr{L}}\rangle \, \mathrm{Tr}\left[B^{\sigma_1} B^{\sigma_2} \ldots B^{\sigma_{\mathscr{L}}}\right]$ with $\sigma_\ell \in \{+1, 0, -1\}$ (12)

$$B^{+1} = \frac{2}{\sqrt{3}} \tau^+ , \quad B^0 = -\frac{2}{\sqrt{3}} \tau^z , \quad B^{-1} = -\frac{2}{\sqrt{3}} \tau^- \qquad (13)$$

with Pauli matrices $\tau^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\tau^- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\tau^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (14)

Now, note that $B^{\pm 1} \underbrace{B^0 \ldots B^0}_{\text{string of } B^0} B^{\pm 1} = 0$

for the Pauli matrices, the operation 'raise, do nothing, raise', yields zero (15)

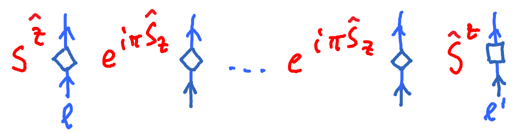
Thus, all 'allowed configurations' (having non-zero coefficients) in AKLT ground state have the property that every ± 1 is followed by string of 0, then ∓ 1.

Allowed: $|\vec{\sigma}_{\mathscr{L}}\rangle = \ldots \; 1 \; 0 \; 0 \; 0 \; -1 \; 0 \; 1 \; 0 \; 0 \; 0 \; 0 \; -1 \; 1 \; 0 \; 0 \; -1$ (16)

Not allowed: $|\vec{\sigma}_{\mathscr{L}}\rangle = \ldots \; \underline{1} \; 0 \; 0 \; 0 \; \underline{1} \; 0 \; 1$ or $0 \; 0 \; \underline{-1} \; 0 \; \underline{-1} \; 1 \; 0$ (17)

'String order parameter' detects this property:

$$\hat{O}^{\text{string}}_{\ell\ell'} \equiv \hat{S}^z_\ell \prod_{\bar{\ell}=\ell+1}^{\ell'-1} e^{i\pi \hat{S}^z_{\bar{\ell}}} \hat{S}^z_{\ell'} =$$



(18)

Eigenvalues of phase factor: $\langle \sigma_\ell | e^{i\pi \hat{S}^z_\ell} | \sigma_\ell \rangle = \begin{cases} -1 & \text{if } \sigma_\ell = \pm 1 \\ +1 & \text{if } \sigma_\ell = 0 \end{cases}$ (19)

Exercise:

Show that the ground state expectation value of string order parameter is non-zero:

$$\lim_{\ell'-\ell \to \infty} \; \lim_{\mathscr{L} \to \infty} \langle g | \hat{O}^{\text{string}}_{\ell\ell'} | g \rangle = -\frac{4}{9} \qquad (20)$$

Hint: first compute $T_{e^{i\pi S_z}}$ (21)

Examples of configurations with $\psi^{\vec{\sigma}} \neq 0$

$+1 \; 0 \; 0 \; -1 \; 0 \; +1 \; 0 \; -1 \; 0 \; +1$

$-1 \; 0 \; 0 \; 1 \; +0 \; -1 \; 0 \; +1 \; 0 \; -1$

Intuitive explanation why string order parameter is nonzero:

$$|g\rangle = \sum_{\vec{\sigma}_{\mathscr{L}}} |\vec{\sigma}_{\mathscr{L}}\rangle \, \psi^{\vec{\sigma}} \qquad (22)$$

$\ell'-1$. z

$$|g\rangle = \sum_{\vec{\sigma}} |\vec{\sigma}\rangle \psi^{\vec{\sigma}} \qquad (22)$$

$$C^{\text{string}}_{\ell\ell'} = \sum_{\vec{\sigma}} |\psi^{\vec{\sigma}}|^2 \langle \vec{\sigma}| \hat{S}^z_\ell \, e^{i\pi \sum_{\bar{\ell}=\ell+1}^{\ell'-1} \hat{S}^z_{\bar{\ell}}} \, \hat{S}^z_{\ell'} |\vec{\sigma}\rangle \qquad (23)$$

For the AKLT ground state, there are six types of configurations; four of them give -1, the other two give 0:

Example configurations	$\langle\vec{\sigma}\vert\hat{S}^z_\ell\vert\vec{\sigma}\rangle$		$\langle\vec{\sigma}\vert e^{i\pi\sum_{\bar{\ell}=\ell+1}^{\ell'-1}\hat{S}^z_{\bar{\ell}}}\vert\vec{\sigma}\rangle$		$\langle\vec{\sigma}\vert\hat{S}^z_{\ell'}\vert\vec{\sigma}\rangle$			(24)
+1 0 0 -1 0 1 0 -1 0 1 :	(+1)	·	(-1)(1)(-1)	·	(+1)	=	-1	(a)
-1 0 0 1 0 -1 0 1 0 -1 :	(-1)	·	(1)(-1)(1)	·	(-1)	=	-1	(b)
+1 0 -1 0 0 1 -1 1 0 -1 :	(+1)	·	(-1)(1)(-1)(1)	·	(-1)	=	-1	(c)
-1 0 1 0 -1 0 1 0 -1 1 :	(-1)	·	(1)(-1)(1)(-1)	·	(+1)	=	-1	(d)
0 -1 1 0 -1 1 0 -1 0 1 :	0					=	0	(e)
1 0 -1 0 1 -1 0 0 1 0 :					0	=	0	(f)

$$C^{\text{string}}_{\ell\ell'} = (-1)\cdot\left(\frac{2}{3}\right)\cdot\left(\frac{2}{3}\right) = -\frac{4}{9}. \qquad (25)$$

probability to get 1 or -1 but not 0 at site ℓ

probability to get 1 or -1 but not 0 at site ℓ'