II: Numerical linear algebra

Within linear algebra, we perform arithmetic operations on finite precision arithmetic.

In the following we consider a finite dim. vector space over real numbers $V_{\mathbb{R}}^n$ $(n \in \mathbb{N})$.

The complex case is obtained from $n \to 2n$.

Available, elementary arithmetic operations:

$+, -$: n-digit arguments typically $\sim O(n)$ complexity

$\cdot, /$: n-digit arguments typically $\sim O(n^2)$ complexity

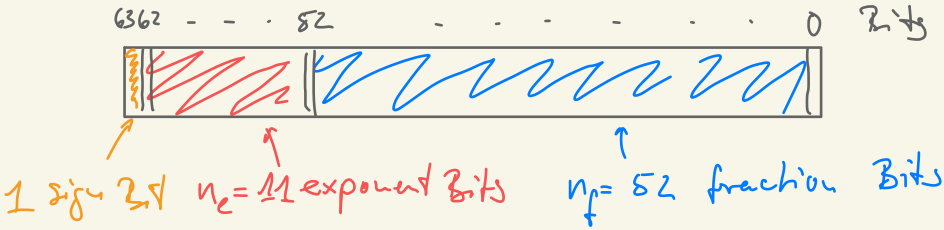
Take home message:

Complexity is estimated w.r.t. # multiplications/divisions!

II.1: Finite precision arithmetics

In finite precision arithmetics there always is a discretization Δ in the representation.

Example: Double precision (64 bit float)



Representation of real number x:

$$x = (-1)^{sign} \cdot \left(1 + \sum_{n=1}^{n_f} x_{n_f-n} \cdot 2^{-n}\right) \cdot 2^{e-1023}$$

with $e = \sum_{n=0}^{n_e-1} x_{n_f+n_e-n}\, 2^n$ the exponent.

Thus here: $\Delta = 2^{-52} \cdot 2^{e-1023}$

Define relative precision: $\delta = 2^{-n_f}$

Thus here: $\delta = 2^{-52} = (1024)^{-5} \cdot \frac{1}{4} \approx 10^{-16}$

Consequences:

(i) $x \in \mathbb{R}$ only represented modulo δ:

$$\tilde{x} = x(1+\delta_x), \quad |\delta_x| \leq \delta/2$$

(ii) $z = x + y$ with rounding errors δ_x, δ_y:

$$\tilde{z} = \tilde{x} + \tilde{y} = z + \underbrace{(x\delta_x + y\delta_y)}_{z\,\delta_z}, \quad |\delta_z| \leq \delta/2$$

(iii) $z = x - y$ with rounding errors δ_x, δ_y:

$$\tilde{z} = \tilde{x} - \tilde{y} = z + (x\delta_x - y\delta_y) = z\Big(1 + \underbrace{\frac{x\delta_x - y\delta_y}{z}}_{\delta_z}\Big)$$

Note: δ_z here not bounded but depends on z! $z \to 0$, then δ_z can explode!

called: Catastrophic cancellation

(iv) $z = x \cdot y$ with rounding errors δ_x, δ_y:

$$\tilde{z} = \tilde{x} \cdot \tilde{y} = x \cdot y\,(1 + \delta_x + \delta_y + \delta_x \cdot \delta_y) = z\,(1 + \delta_z)$$

note that if $\tilde{x} \cdot \tilde{y} < \delta$, then $\delta_z > \delta$ possible.

To see this write

$$\tilde{x} = (1 + r_x)\,2^{e_x - 1023}, \quad \tilde{y} = (1 + r_y)\,2^{e_y - 1023}, \quad |r_{x,y}| < 1$$

$$\Rightarrow \tilde{x} \cdot \tilde{y} = (1 + r_x)(1 + r_y)\,2^{e_x + e_y - 2046}$$

consider $\underbrace{r_x r_y}\, 2^{e_x + e_y - 2046}$

$$= \sum_{n,m=0}^{n_f} \left(x_{n_f - n}\, y_{n_f - m}\, 2^{-(n+m)} \right)$$

and thus: $$|\delta_z| = \sum_{n+m > n_f}^{n_f} x_{n_f - n}\, y_{n_f - m}\, 2^{-(n+m)}$$

$\Rightarrow$ multiplication suppresses relevant bits

(v) $z = x/y$ similar to $x \cdot y$ but here errors are magnified (can be worked out similar to (iv))

Keep these rounding errors in mind as they can drastically impact outcome!

Example:

(i) $$f(x) = \frac{x - \sin(x)}{1 - \cos(x)} \equiv y$$

Taylor expand $\sin(x)$ & $\cos(x)$ & evaluate at finite precision arithmetics for $x \ll 1$

$$\hat{y} = \frac{x(\delta_x - \delta_x)}{x^2(1-\delta_x)^2}$$

enumerator evaluates with precision δ but denominator with max precision $\sqrt{\delta}$!

$\Rightarrow x < \sqrt{\delta}$ then y undefined

(ii) $\mathrm{Var}(X) = \langle (X - \langle X \rangle)^2 \rangle$

evaluating $Var(X) = \langle X^2 \rangle - \langle X \rangle^2$
limits precision to $\delta Var(X) \sim \sqrt{\delta}$ (catastrophic cancellation)

(iii) $z = x^2 - y^2$

evaluating $z = (x^2) - (y^2)$ limits precision to $\sqrt{\delta}$. But evaluating $z = (x+y)(x-y)$ evaluates to precision δ !

II. 2 Matrix-Matrix multiplication

In the following, we consider for simplicity quadr. matrices $\in V_{\mathbb{R}}^{m \times m}$ with $m = 2^n$, $n \in \mathbb{N}$.

Divide and conquer

Decompose $\underline{\underline{C}} = \underline{\underline{A}} \cdot \underline{\underline{B}}$ with $\underline{\underline{A}}, \underline{\underline{B}} \in V_{\mathbb{R}}^{m \times m}$ into

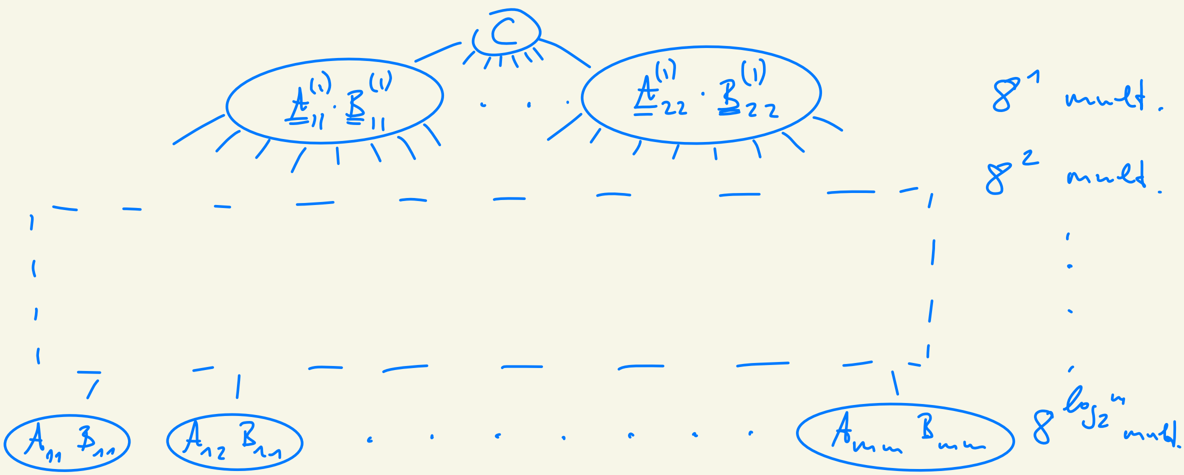
$$\begin{pmatrix} \underline{\underline{C}}_{11} & \underline{\underline{C}}_{12} \\ \underline{\underline{C}}_{21} & \underline{\underline{C}}_{22} \end{pmatrix} = \begin{pmatrix} \underline{\underline{A}}_{11} & \underline{\underline{A}}_{12} \\ \underline{\underline{A}}_{21} & \underline{\underline{A}}_{22} \end{pmatrix} \cdot \begin{pmatrix} \underline{\underline{B}}_{11} & \underline{\underline{B}}_{12} \\ \underline{\underline{B}}_{21} & \underline{\underline{B}}_{22} \end{pmatrix}$$

with $\underline{\underline{A}}_{ij}, \underline{\underline{B}}_{ij}, \underline{\underline{C}}_{ij} \in V_{\mathbb{R}}^{m/2 \times m/2}$ and

$$
\left.
\begin{aligned}
\underline{\underline{C}}_{11} &= \underline{\underline{A}}_{11}\cdot \underline{\underline{B}}_{11} + \underline{\underline{A}}_{12}\cdot \underline{\underline{B}}_{21}\\
\underline{\underline{C}}_{12} &= \underline{\underline{A}}_{11}\cdot \underline{\underline{B}}_{21} + \underline{\underline{A}}_{12}\cdot \underline{\underline{B}}_{22}\\
\underline{\underline{C}}_{21} &= \underline{\underline{A}}_{21}\cdot \underline{\underline{B}}_{11} + \underline{\underline{A}}_{22}\cdot \underline{\underline{B}}_{21}\\
\underline{\underline{C}}_{22} &= \underline{\underline{A}}_{21}\cdot \underline{\underline{B}}_{12} + \underline{\underline{A}}_{22}\cdot \underline{\underline{B}}_{22}
\end{aligned}
\right\}\ 8 \text{ M-M-operations}
$$

For each M-M-operation $\underline{\underline{A}}_{ij}\cdot\underline{\underline{B}}_{kl}$, $i,j,k,l \in \{1,2\}$ repeat scheme. After $\log_2 n$ recursions we arrive at 8 simple multiplications.

The total number of multiplications can be deduced from drawing iteration tree:



$$\Rightarrow \#\,\text{multiplications} = 8^{\log_2 n} = (2^3)^{\log_2 n} = n^3$$

Even though that is not suprising, it tells us two important things:

(i) M-M products can be parallelized systematically

(ii) The counting technique suggests investigating the decomposition for more efficient scheme

Indeed, the 8 equations are highly symmetric!

Strassen's algorithm (1969)

There are various ways to prove Strassen's algorithm. Let us begin by introducing a basis for $V_{\mathbb{R}}^{2\times 2}$

$$\underline{\underline{e}}^1 = |\underline{e}^1\rangle\langle\underline{e}^1| = \begin{pmatrix}1&0\\0&0\end{pmatrix}, \quad \underline{\underline{e}}^2 = |\underline{e}^1\rangle\langle\underline{e}^2| = \begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$\underline{\underline{e}}^3 = |\underline{e}^2\rangle\langle\underline{e}^1| = \begin{pmatrix}0&0\\1&0\end{pmatrix}, \quad \underline{\underline{e}}^4 = |\underline{e}^2\rangle\langle\underline{e}^2| = \begin{pmatrix}0&0\\0&1\end{pmatrix}$$

with $|\underline{e}^1\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix}$, $|\underline{e}^2\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix}$

They obey the algebra:

$$\left.\begin{array}{llll}
\underline{\underline{e}}^1\cdot\underline{\underline{e}}^1 = \underline{\underline{e}}^1 & \underline{\underline{e}}^2\cdot\underline{\underline{e}}^1 = 0 & \underline{\underline{e}}^3\cdot\underline{\underline{e}}^1 = \underline{\underline{e}}^3 & \underline{\underline{e}}^4\cdot\underline{\underline{e}}^1 = 0 \\
\underline{\underline{e}}^1\cdot\underline{\underline{e}}^2 = \underline{\underline{e}}^2 & \underline{\underline{e}}^2\cdot\underline{\underline{e}}^2 = 0 & \underline{\underline{e}}^3\cdot\underline{\underline{e}}^2 = \underline{\underline{e}}^4 & \underline{\underline{e}}^4\cdot\underline{\underline{e}}^2 = 0 \\
\underline{\underline{e}}^1\cdot\underline{\underline{e}}^3 = 0 & \underline{\underline{e}}^2\cdot\underline{\underline{e}}^3 = \underline{\underline{e}}^1 & \underline{\underline{e}}^3\cdot\underline{\underline{e}}^3 = 0 & \underline{\underline{e}}^4\cdot\underline{\underline{e}}^3 = \underline{\underline{e}}^3 \\
\underline{\underline{e}}^1\cdot\underline{\underline{e}}^4 = 0 & \underline{\underline{e}}^2\cdot\underline{\underline{e}}^4 = \underline{\underline{e}}^2 & \underline{\underline{e}}^3\cdot\underline{\underline{e}}^4 = 0 & \underline{\underline{e}}^4\cdot\underline{\underline{e}}^4 = \underline{\underline{e}}^4
\end{array}\right\} (*)$$

Now, we can write for any $\underline{\underline{A}} \in V_{\mathbb{R}}^{m \times m}$

$$\underline{\underline{A}} = \underline{\underline{A}}_{11} \otimes \underline{\underline{e}}^1 + \underline{\underline{A}}_{12} \otimes \underline{\underline{e}}^2 + \underline{\underline{A}}_{21} \otimes \underline{\underline{e}}^3 + \underline{\underline{A}}_{22} \otimes \underline{\underline{e}}^4$$

$$= \underline{\underline{A}}_1 \otimes \underline{\underline{e}}^1 + \underline{\underline{A}}_2 \otimes \underline{\underline{e}}^2 + \underline{\underline{A}}_3 \otimes \underline{\underline{e}}^3 + \underline{\underline{A}}_4 \otimes \underline{\underline{e}}^4$$

Note that

$$\underline{\underline{A}} + \lambda \underline{\underline{B}} = (\underline{\underline{A}}_1 + \lambda \underline{\underline{B}}_1) \otimes \underline{\underline{e}}^1 + (\underline{\underline{A}}_2 + \lambda \underline{\underline{B}}_2) \otimes \underline{\underline{e}}^2 + \cdots$$

$$= \sum_{j=1}^{4} (\underline{\underline{A}}_j + \lambda \underline{\underline{B}}_j) \otimes \underline{\underline{e}}^j$$

We can thus interpret any $\underline{\underline{A}} \in V_{\mathbb{R}}^{m \times m}$ as linear combination of states formed from map "$\otimes$" via:

$$\otimes : V_{\mathbb{R}}^{m/2 \times m/2} \times V_{\mathbb{R}}^{2 \times 2} \to V_{\mathbb{R}}^{m \times m}.$$

We now expand $\underline{\underline{C}} = \underline{\underline{A}} \cdot \underline{\underline{B}}$ using different basis sets $\{\underline{\underline{b}}^j\}, \{\underline{\underline{\tilde{b}}}^j\}$

$$\underline{\underline{C}} = \sum_{i,j=1}^{4} (\underline{\underline{A}}_i \cdot \underline{\underline{B}}_j) \otimes (\underline{\underline{b}}^i \cdot \underline{\underline{\tilde{b}}}^j)$$

$$= (\underline{\underline{A}}_1 \cdot \underline{\underline{B}}_1) \otimes (\underline{\underline{b}}^1 \cdot \underline{\underline{\tilde{b}}}^1) + (\underline{\underline{A}}_1 \cdot \underline{\underline{B}}_2) \otimes (\underline{\underline{b}}^1 \cdot \underline{\underline{\tilde{b}}}^2) + (\underline{\underline{A}}_1 \cdot \underline{\underline{B}}_3) \otimes (\underline{\underline{b}}^1 \cdot \underline{\underline{\tilde{b}}}^3) + (\underline{\underline{A}}^1 \cdot \underline{\underline{B}}^4) \otimes (\underline{\underline{b}}^1 \cdot \underline{\underline{\tilde{b}}}^4)$$

$$+ (\underline{\underline{A}}_2 \cdot \underline{\underline{B}}_1) \otimes (\underline{\underline{b}}^2 \cdot \underline{\underline{\tilde{b}}}^1) + \cdots$$

$$\vdots$$

$$+ (\underline{\underline{A}}_4 \cdot \underline{\underline{B}}_1) \otimes (\underline{\underline{b}}^4 \cdot \underline{\underline{\tilde{b}}}^1) + \cdots + (\underline{\underline{A}}_4 \cdot \underline{\underline{B}}_4) \otimes (\underline{\underline{b}}^4 \cdot \underline{\underline{\tilde{b}}}^4)$$

If we would choose for the canonical basis sets $\underline{\underline{b}}^j = \underline{\underline{\tilde{b}}}^j = \underline{\underline{e}}^j$, then using (*) we would arrive at

$$\underline{\underline{C}} = (\underline{\underline{A}}_1 \cdot \underline{\underline{B}}_1) \otimes \underline{\underline{e}}^1 + (\underline{\underline{A}}_1 \cdot \underline{\underline{B}}_2) \otimes \underline{\underline{e}}^2$$
$$+ (\underline{\underline{A}}_2 \cdot \underline{\underline{B}}_3) \otimes \underline{\underline{e}}^1 + (\underline{\underline{A}}_2 \cdot \underline{\underline{B}}_4) \otimes \underline{\underline{e}}^2$$
$$+ (\underline{\underline{A}}_3 \cdot \underline{\underline{B}}_1) \otimes \underline{\underline{e}}^3 + (\underline{\underline{A}}_3 \cdot \underline{\underline{B}}_2) \otimes \underline{\underline{e}}^4$$
$$+ (\underline{\underline{A}}_4 \cdot \underline{\underline{B}}_3) \otimes \underline{\underline{e}}^3 + (\underline{\underline{A}}_4 \cdot \underline{\underline{B}}_4) \otimes \underline{\underline{e}}^4$$

which is just the result from the divide-and-conquer decomposition.

Thus, the question reduces to:

Can we find basis sets $\underline{\underline{b}}^j, \underline{\underline{\tilde{b}}}^j$ such that we need less multiplications?

Let us introduce: $\underline{\underline{D}} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \underline{\underline{e}}^3 - (\underline{\underline{e}}^2 + \underline{\underline{e}}^4)$.

Note that $\underline{\underline{D}}$ is rotation matrix & $\underline{\underline{D}}^3 = \underline{\underline{1}} \Rightarrow \underline{\underline{D}}^2 = \underline{\underline{D}}^{-1}$

Then choose $\underline{\underline{X}} = \underline{\underline{e}}^2$ (important: $\underline{\underline{D}}\,\underline{\underline{X}} \neq \underline{\underline{X}}$)

we have:

$$\underline{\underline{D}}^{-1} = \underline{\underline{D}}^2 = \underline{\underline{e}}^2 - \underline{\underline{e}}^1 - \underline{\underline{e}}^3$$
$$\underline{\underline{D}}\,\underline{\underline{X}}\,\underline{\underline{D}} = \underline{\underline{e}}^3 - \underline{\underline{e}}^4$$

and :

$$\underline{\underline{X}}\ \underline{\underline{1}}\ \underline{\underline{X}} = \underline{\underline{e}}^2 \cdot \underline{\underline{e}}^2 = 0$$

$$\underline{\underline{X}}\ \underline{\underline{D}}\ \underline{\underline{X}} = \underline{\underline{e}}^2 (\underline{\underline{e}}^3 - \underline{\underline{e}}^2 - \underline{\underline{e}}^4)\, \underline{\underline{e}}^2 = (\underline{\underline{e}}^1 - \underline{\underline{e}}^2)\, \underline{\underline{e}}^2 = \underline{\underline{e}}^2 = \underline{\underline{X}}$$

$$\underline{\underline{X}}\ \underline{\underline{D}}^2\ \underline{\underline{X}} = (\underline{\underline{e}}^1 - \underline{\underline{e}}^2)\, \underline{\underline{e}}^4 = -\underline{\underline{e}}^2 = -\underline{\underline{X}}$$

it can be shown that the two sets are a basis of $V_{\mathbb{R}}^{2\times 2}$:

$$\mathcal{B}_1 = \{\underline{\underline{D}},\ \underline{\underline{X}},\ \underline{\underline{D}}^{-1}\underline{\underline{X}}\,\underline{\underline{D}},\ \underline{\underline{D}}\,\underline{\underline{X}}\,\underline{\underline{D}}^{-1}\} = \{\underline{\underline{b}}^1, \underline{\underline{b}}^2, \underline{\underline{b}}^3, \underline{\underline{b}}^4\}$$

$$\mathcal{B}_2 = \{\underline{\underline{D}}^{-1},\ \underline{\underline{X}},\ \underline{\underline{D}}^{-1}\underline{\underline{X}}\,\underline{\underline{D}},\ \underline{\underline{D}}\,\underline{\underline{X}}\,\underline{\underline{D}}^{-1}\} = \{\tilde{\underline{\underline{b}}}^1, \tilde{\underline{\underline{b}}}^2, \tilde{\underline{\underline{b}}}^3, \tilde{\underline{\underline{b}}}^4\}$$

Now we compute all products of $\underline{\underline{b}}^i \in \mathcal{B}_1$ & $\tilde{\underline{\underline{b}}}^j \in \mathcal{B}_2$

$\underline{\underline{b}}^i \backslash \tilde{\underline{\underline{b}}}^j$	$\underline{\underline{D}}^{-1}$	$\underline{\underline{X}}$	$\underline{\underline{D}}^{-1}\underline{\underline{X}}\,\underline{\underline{D}}$	$\underline{\underline{D}}\,\underline{\underline{X}}\,\underline{\underline{D}}^{-1}$
$\underline{\underline{D}}$	$\underline{\underline{1}}$	$\underline{\underline{D}}\,\underline{\underline{X}}$	$\underline{\underline{X}}\,\underline{\underline{D}}$	$\underline{\underline{D}}^{-1}\underline{\underline{X}}\,\underline{\underline{D}}^{-1}$
$\underline{\underline{X}}$	$\underline{\underline{X}}\,\underline{\underline{D}}^{-1}$	$\underline{\underline{0}}$	$-\underline{\underline{X}}\,\underline{\underline{D}}$	$\underline{\underline{X}}\,\underline{\underline{D}}^{-1}$
$\underline{\underline{D}}^{-1}\underline{\underline{X}}\,\underline{\underline{D}}$	$\underline{\underline{D}}^{-1}\underline{\underline{X}}$	$\underline{\underline{D}}^{-1}\underline{\underline{X}}$	$\underline{\underline{0}}$	$-\underline{\underline{D}}^{-1}\underline{\underline{X}}\,\underline{\underline{D}}^{-1}$
$\underline{\underline{D}}\,\underline{\underline{X}}\,\underline{\underline{D}}^{-1}$	$\underline{\underline{D}}\,\underline{\underline{X}}\,\underline{\underline{D}}$	$-\underline{\underline{D}}\,\underline{\underline{X}}$	$\underline{\underline{D}}\,\underline{\underline{X}}\,\underline{\underline{D}}$	$\underline{\underline{0}}$

Note that only seven products occur!

Thus we can expand $\underline{\underline{C}} = \underline{\underline{A}} \cdot \underline{\underline{B}}$ in this basis using

$$\underline{\underline{b}}^1 \cdot \underline{\underline{\tilde{b}}}^1 = \mathbb{1} = \underline{\underline{e}}^1 + \underline{\underline{e}}^4$$

$$\underline{\underline{b}}^1 \cdot \underline{\underline{\tilde{b}}}^2 = \underline{\underline{D}}\,\underline{\underline{X}} = \underline{\underline{e}}^4, \qquad \underline{\underline{b}}^1 \cdot \underline{\underline{\tilde{b}}}^3 = \underline{\underline{X}}\,\underline{\underline{D}} = \underline{\underline{e}}^1 - \underline{\underline{e}}^2$$

$$\underline{\underline{b}}^2 \cdot \underline{\underline{\tilde{b}}}^1 = \underline{\underline{X}}\,\underline{\underline{D}}^{-1} = -\underline{\underline{e}}^1, \qquad \underline{\underline{b}}^3 \cdot \underline{\underline{\tilde{b}}}^1 = \underline{\underline{D}}^{-1}\underline{\underline{X}} = -\underline{\underline{e}}^2 - \underline{\underline{e}}^4$$

$$\underline{\underline{b}}^1 \cdot \underline{\underline{\tilde{b}}}^4 = \underline{\underline{D}}^{-1}\underline{\underline{X}}\,\underline{\underline{D}}^{-1} = \underline{\underline{e}}^1 + \underline{\underline{e}}^3, \qquad \underline{\underline{b}}^4 \cdot \underline{\underline{\tilde{b}}}^1 = \underline{\underline{D}}\,\underline{\underline{X}}\,\underline{\underline{D}} = \underline{\underline{e}}^3 - \underline{\underline{e}}^4$$

such that

$$\begin{aligned}
\underline{\underline{C}} = & \left(\underline{\underline{A}}_1 \cdot \underline{\underline{B}}_1\right) \otimes \left(\underline{\underline{e}}^1 + \underline{\underline{e}}^4\right) \\
& + \left(\underline{\underline{A}}_1 - \underline{\underline{A}}_4\right) \cdot \underline{\underline{B}}_2 \otimes \underline{\underline{e}}^4 \\
& + \left(\underline{\underline{A}}_1 - \underline{\underline{A}}_2\right) \cdot \underline{\underline{B}}_3 \otimes \left(\underline{\underline{e}}^1 - \underline{\underline{e}}^2\right) \\
& - \underline{\underline{A}}_2\left(\underline{\underline{B}}_1 + \underline{\underline{B}}_4\right) \otimes \underline{\underline{e}}^1 \qquad (**) \\
& - \underline{\underline{A}}_3\left(\underline{\underline{B}}_1 + \underline{\underline{B}}_2\right) \otimes \left(\underline{\underline{e}}^2 + \underline{\underline{e}}^4\right) \\
& + \left(\underline{\underline{A}}_1 - \underline{\underline{A}}_3\right) \underline{\underline{B}}_4 \otimes \left(\underline{\underline{e}}^1 + \underline{\underline{e}}^3\right) \\
& + \underline{\underline{A}}_4\left(\underline{\underline{B}}_1 + \underline{\underline{B}}_3\right) \otimes \left(\underline{\underline{e}}^3 - \underline{\underline{e}}^4\right)
\end{aligned}$$

Now we only need to represent $\underline{\underline{A}}_i$ & $\underline{\underline{B}}_i$ in the old basis $\underline{\underline{e}}^i$:

$$\underline{\underline{b}}^1 = \underline{\underline{e}}^3 - \underline{\underline{e}}^2 - \underline{\underline{e}}^4 \qquad \underline{\underline{\tilde{b}}}^1 = \underline{\underline{e}}^2 - \underline{\underline{e}}^1 - \underline{\underline{e}}^3$$

$$\underline{\underline{b}}^2 = \underline{\underline{e}}^2 \qquad \underline{\underline{\tilde{b}}}^2 = \underline{\underline{e}}^2$$

$\underline{\underline{b}}^3 = -\underline{\underline{e}}^1 + \underline{\underline{e}}^2 - \underline{\underline{e}}^3 + \underline{\underline{e}}^4$

$\underline{\underline{b}}^4 = -\underline{\underline{e}}^3$

$$\Rightarrow \underline{\underline{A}} = \underline{\underline{A}}_1 \otimes (\underline{\underline{e}}^3 - \underline{\underline{e}}^2 - \underline{\underline{e}}^4) + \underline{\underline{A}}_2 \otimes \underline{\underline{e}}^2 + \underline{\underline{A}}_3 \otimes (-\underline{\underline{e}}^1 + \underline{\underline{e}}^2 - \underline{\underline{e}}^3 + \underline{\underline{e}}^4) - \underline{\underline{A}}_4 \otimes \underline{\underline{e}}^3$$

$$\overset{!}{=} \underline{\underline{A}}_{11} \otimes \underline{\underline{e}}^1 + \underline{\underline{A}}_{12} \otimes \underline{\underline{e}}^2 + \underline{\underline{A}}_{21} \otimes \underline{\underline{e}}^3 + \underline{\underline{A}}_{22} \otimes \underline{\underline{e}}^4$$

$\hat{\underline{\underline{b}}}^3 = -\underline{\underline{e}}^1 + \underline{\underline{e}}^2 - \underline{\underline{e}}^3 + \underline{\underline{e}}^4$

$\hat{\underline{\underline{b}}}^4 = -\underline{\underline{e}}^3$

$$\Rightarrow \underline{\underline{B}} = \underline{\underline{B}}_1 \otimes (\underline{\underline{e}}^2 - \underline{\underline{e}}^1 - \underline{\underline{e}}^3) + \underline{\underline{B}}_2 \otimes \underline{\underline{e}}^2 + \underline{\underline{B}}_3 \otimes (-\underline{\underline{e}}^1 + \underline{\underline{e}}^2 - \underline{\underline{e}}^3 + \underline{\underline{e}}^4) - \underline{\underline{B}}_4 \otimes \underline{\underline{e}}^3$$

$$\overset{!}{=} \underline{\underline{B}}_{11} \otimes \underline{\underline{e}}^1 + \underline{\underline{B}}_{12} \otimes \underline{\underline{e}}^2 + \underline{\underline{B}}_{21} \otimes \underline{\underline{e}}^3 + \underline{\underline{B}}_{22} \otimes \underline{\underline{e}}^4$$

solving for $\underline{\underline{A}}_i (\underline{\underline{A}}_{11}, \dots, \underline{\underline{A}}_{22})$ & $\underline{\underline{B}}_i (\underline{\underline{B}}_{11}, \dots, \underline{\underline{B}}_{22})$ and inserting into (**), after some further algebra one arrives at

$$\underline{\underline{C}} = (\underline{\underline{M}}_{I} + \underline{\underline{M}}_{IV} - \underline{\underline{M}}_{V} + \underline{\underline{M}}_{VII}) \otimes \underline{\underline{e}}^1 + (\underline{\underline{M}}_{III} + \underline{\underline{M}}_{V}) \otimes \underline{\underline{e}}^2 + (\underline{\underline{M}}_{II} + \underline{\underline{M}}_{IV}) \otimes \underline{\underline{e}}^3 + (\underline{\underline{M}}_{I} + \underline{\underline{M}}_{III} - \underline{\underline{M}}_{II} + \underline{\underline{M}}_{VI}) \otimes \underline{\underline{e}}^4$$

with

$\underline{\underline{M}}_{I} = (\underline{\underline{A}}_{11} + \underline{\underline{A}}_{22})(\underline{\underline{B}}_{11} + \underline{\underline{B}}_{22})$, $\underline{\underline{M}}_{II} = (\underline{\underline{A}}_{11} + \underline{\underline{A}}_{22})\underline{\underline{B}}_{11}$

$\underline{\underline{M}}_{III} = \underline{\underline{A}}_{11}(\underline{\underline{B}}_{12} - \underline{\underline{B}}_{22})$, $\underline{\underline{M}}_{IV} = \underline{\underline{A}}_{22}(-\underline{\underline{B}}_{11} + \underline{\underline{B}}_{21})$, $\underline{\underline{M}}_{V} = (\underline{\underline{A}}_{11} + \underline{\underline{A}}_{12})\underline{\underline{B}}_{22}$

$\underline{\underline{M}}_{VI} = (-\underline{\underline{A}}_{11} + \underline{\underline{A}}_{21})(\underline{\underline{B}}_{11} + \underline{\underline{B}}_{12})$, $\underline{\underline{M}}_{VII} = (\underline{\underline{A}}_{12} - \underline{\underline{A}}_{22})(\underline{\underline{B}}_{21} + \underline{\underline{B}}_{22})$

We can analyze the complexity by drawing the same tree of iterations but now with only 7 multiplications, i.e. 7 branches emerging from each node.

$\Rightarrow$ # multiplications $= 7^{\log_2 n} = n^{\log_2 7}$!!!