Lecture II - Bosonization Identity

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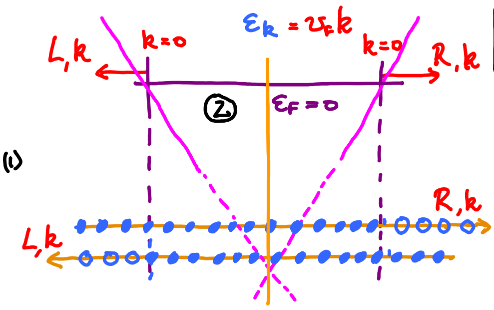
Summary: lecture I - fermions

$$\psi_{\text{phys}}(x) \;\text{"="}\; e^{-ip_F x}\,\psi_L(x) + e^{ip_F x}\,\psi_R(-x) \qquad (1)$$



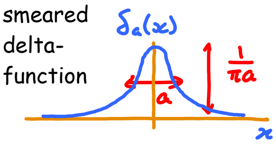
$$\psi_\eta(x) = \Delta^{1/2} \sum_k e^{-|k|a}\, e^{-ikx}\, c_{k\eta}\,, \qquad (2)$$

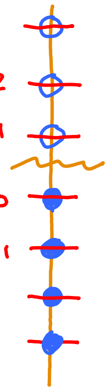
$$\Delta_L = \frac{2\pi}{L}\,, \quad k = \Delta_L(n - 1/2)\,, \quad n \in \mathbb{Z} \qquad (3)$$

$$\{c_{k\eta}, c^\dagger_{k'\eta'}\} = \delta_{kk'}\,\delta_{\eta\eta'} \qquad (4)$$

$$\{\psi_\eta(x), \psi^\dagger_{\eta'}(x')\} = \delta_{\eta\eta'}\, 2\pi\, \delta_a(x - x') \qquad (5)$$

smeared delta-function





$$H = \sum_{k\eta}{}' \, v_F k\, c^\dagger_{k\eta} c_{k\eta} \qquad (6)$$

$$c^\dagger_{k\eta}|0\rangle = 0, \qquad c_{k\eta}|0\rangle = 0 \qquad (7)$$

$z = \tau + ix$

$$\langle T\, \psi_\eta(z)\, \psi^\dagger_{\eta'}(0)\rangle = \frac{\delta_{\eta\eta'}\, e^{\pi(\sigma+1)/L}}{\frac{L}{\pi}\sinh\left[\frac{\pi}{L}(z + \sigma a)\right]} \xrightarrow{L\to\infty} \delta_{\eta\eta'}\, \frac{1}{z + \sigma a} \qquad (8)$$

Summary: lecture I - bosons

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$$\hat N_\eta = \sum_k :c^\dagger_{k\eta} c_{k\eta}: \qquad (1)$$

$\underline{\underline{q > 0}}$

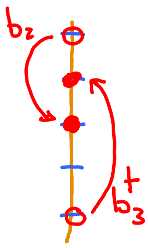
$$[N_\eta, b^\dagger_{q\eta}] = [N_\eta, b_{q\eta}] = 0 \qquad (4)$$

$$b_{q\eta} = \frac{-i}{\sqrt{n_q}} \sum_k c^\dagger_{k-q,\eta}\, c_{k\eta} \qquad (2)$$

$$[b_{q\eta}, b_{q'\eta'}] = [b_{q\eta}, b^\dagger_{q'\eta'}] = 0 \qquad (5)$$

$$b^\dagger_{q\eta} = \frac{i}{\sqrt{n_q}} \sum_k c^\dagger_{k+q,\eta}\, c_{k\eta} \qquad (3)$$

$$[b_{q\eta}, b^\dagger_{q'\eta'}] = \delta_{qq'}\,\delta_{\eta\eta'} \qquad (6)$$



$$\varphi_\eta(x) = -\sum_{q>0} e^{-aq/2}\, \frac{1}{\sqrt{n_q}}\, e^{-iqx}\, b_{q\eta}\,, \qquad (7)$$

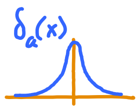
$$q = \Delta_L n_q\,, \quad 0 < n_q \in \mathbb{Z}^+ \qquad (8)$$

$$\phi_\eta(x) = \varphi_\eta(x) + \varphi^\dagger_\eta(x)\,, \qquad (9)$$

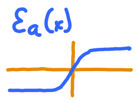
$$\rho_\eta(x) = :\psi^\dagger_\eta(x)\,\psi_\eta(x): \; = \Delta_L N_\eta + \partial_x \phi \qquad (10)$$

$$[\varphi(x), \varphi^\dagger(x')] = \ln[i\Delta_L(x - x' - ia)] \qquad (11)$$

$$[\phi(x), \partial_{x'}\phi(x')] = 2\pi i \left[\delta_a(x) - \frac{1}{L}\right] \qquad (12)$$



$$[\phi(x), \phi(x')] = -i\pi\, \varepsilon_a(x - x') \qquad (13)$$



Derivation of bosonization identity

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Outline

1. Bosonic reorganization of Fock space $\quad |\{n_{k\eta}\}\in\{0,1\}\rangle \Leftrightarrow |N, \{m_{q\eta}\}\in 0,1,\dots\rangle$

2. Number ladder operators - "Klein factors" F $\quad F^\dagger|N\rangle = |N+1\rangle$, $\quad F|N\rangle = |N-1\rangle$

3. Bosonic coherent states $\quad b_q \psi |N\rangle_0 = \alpha_q \psi |N\rangle_0$

5. Action of fermion field on N-particle ground state $\quad \psi|N\rangle_0 = \underbrace{e^{\sum_q \alpha_q b_q^\dagger}}_{= e^{-i\varphi^\dagger}} F\lambda|N\rangle$

6. Action of fermion field on general N-particle state $\quad \psi|N\rangle = F\lambda e^{-i\varphi} e^{-i\varphi^\dagger}|N\rangle$

holds for general state, hence we have operator identity $\quad \boxed{\psi = F\lambda e^{-i\varphi} e^{-i\varphi^\dagger}}$

II.1 Bosonic reorganization of Fock space

$\eta = 1, \dots, M$

Fermionic Fock space for M species: $\mathcal{F} = \sum_{\oplus \vec{N}} \mathcal{H}_{\vec{N}}$, Hilbert space with fixed particle number $\vec{N}$ $(N_1, \dots, N_M)$ (1)

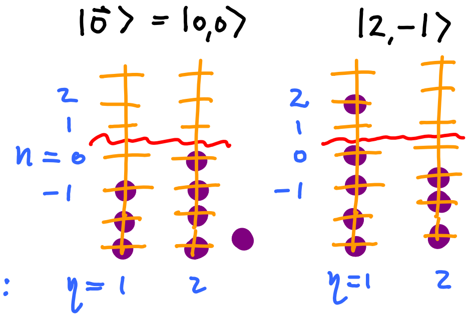
$L\uparrow, L\downarrow, R\uparrow, R\downarrow \Rightarrow M = 4$

$$|\vec{N}\rangle_0 = |N_1, \dots, N_M\rangle = (c_1^\dagger)^{N_1}(c_2^\dagger)^{N_2}\dots(c_M^\dagger)^{N_M}|\vec{0}\rangle \quad (2)$$

Fermi ground state

N-particle ground state:

$$\begin{cases} c^\dagger_{N_\eta \eta}\, c^\dagger_{N_\eta-1,\eta} \cdots c^\dagger_{1\eta} & N_\eta > 0 \\ 1 & \text{for } N_\eta = 0 \\ c_{N_\eta+1,\eta}\, c_{N_\eta+2,\eta} \cdots c_{0\eta} & N_\eta < 0 \end{cases} \quad (3)$$

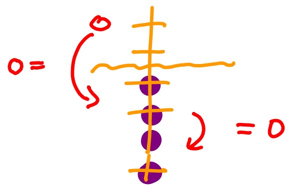


$|\vec{N}\rangle_0$

Each has no particle-hole excitations, and hence acts as vacuum-state for boson operators:

$$b_{q\eta}|\vec{N}\rangle_0 = 0 \quad (4)$$

$b_{q\eta} \sim \sum_k c^\dagger_{k-q} c_k$



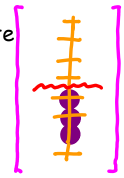
Completeness

Every N-particle state can be written i.t.o. p-h excitations acting on N-particle ground state

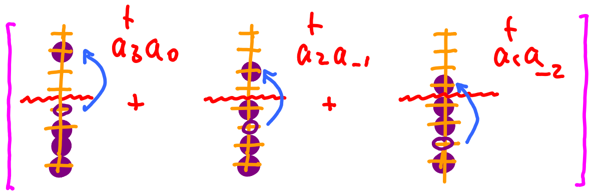
$$= f(b^\dagger)\,|\vec N\rangle_0 \qquad (2)$$

Example for M=1:

$$b^\dagger_3\,|0\rangle_0 = \left(\frac{1}{\sqrt3}\sum_n a^\dagger_{n+3}\,a_n\right)$$



$$= \frac{1}{\sqrt3}\Big[\ a^\dagger_3 a_0 \ + \ a^\dagger_2 a_{-1} \ + \ a^\dagger_1 a_{-2}\ \Big]$$



$b^\dagger_q$ ● $c^\dagger c$, and "vice versa".

Intuitive reason for (2): ~~is a linear combination of~~

$$\boxed{b^\dagger\text{'s, acting on } |\vec N\rangle_0, \text{ span } \mathcal{H}_{\vec N}\ !!} \qquad (3)$$

~~Conclusion:~~

$$= \text{span}\left\{|\vec N, \{m_{q\eta}\}\rangle,\ N_\eta\in\mathbb{Z},\ m_{q\eta}\in\mathbb{N}_+\right\}, \qquad |\vec N,\{m_{q\eta}\}\rangle = \prod_{q\eta}\frac{1}{(m_{q\eta}!)^{1/2}}\left(b^\dagger_{q\eta}\right)^{m_{q\eta}}|\vec N\rangle_0 \qquad (4)$$

$$= \text{span}\left\{|\{n_{k\eta}\}\rangle,\ n_{k\eta}\in\{0,1\}\right\} \qquad (5)$$

$$\text{Tr}_f\left[e^{-\beta\hat H_f}\right] = Z_f = Z_b = \text{Tr}_b\left[e^{-\beta\hat H_b}\right] \qquad (6)$$

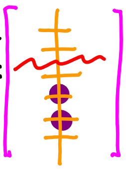
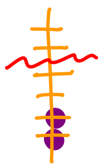
Formal proof: Haldane, 1981; vDS, App. B

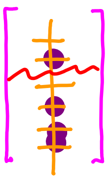
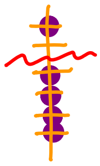
II.2 Number ladder operators - "Klein factors" $\left\{ {F^\dagger_\eta \atop F_\eta} \right\}: N_\eta \to N_\eta \pm 1$ (1)

"Klein factor" changes ... without affecting structure of particle-hole excitations!

$$F\,|-1\rangle = |-2\rangle \qquad F^\dagger_1\, b^\dagger_1\,|0\rangle_0 = b^\dagger_1\, F^\dagger_1\,|0\rangle_0 \qquad (2)$$

Example for M=1:

F  = 

$F^\dagger_1$  = 

Definition of action of F on arbitrary state in ●:

$$\left\{ {F^\dagger_\eta \atop F_\eta} \right\}|\vec N\rangle = \left\{ {F^\dagger_\eta \atop F_\eta} \right\} f(b^\dagger)\,|\vec N\rangle := f(b^\dagger)\, c^\dagger_{N_\eta \pm 1}\,|\vec N\rangle_0 = f(b^\dagger)\,\hat T_\eta\,|\vec N \pm 1_\eta\rangle_0 \qquad (3)$$

commutes by definition

$(\dots 0, 1, 0 \dots)$

$$\underbrace{(c_1)^{N_1}\dots(c_\eta)^{N_\eta}\dots(c_M)^{N_M}\,|\vec 0\rangle}_{(4.2)}$$

$$\underbrace{(c^\dagger_{N_\eta\eta}\dots c^\dagger_{1\eta})}_{(4.3)}$$

$$\hat T_\eta = (-1)^{\sum_{\eta'=1}^{\eta-1}\hat N_{\eta'}} \qquad (4)$$

fermionic phase factor:

$$\left\{ {F^\dagger_\eta \atop F_\eta} \right\} = \sum_{\vec N}\sum_{\{m_q\}} \hat T_\eta\,|\vec N \pm 1_\eta; \{m_{q\eta}\}\rangle\langle \vec N; \{m_{q\eta}\}| \qquad (5)$$

Equivalent definition:

Commutation relations involving Klein factors

Klein factors do not affect p-h structure:

$$[F_\eta, b_{q\eta'}] = [F_\eta, b^\dagger_{q\eta'}] = [F^\dagger_\eta, b_{q\eta'}] = [F^\dagger_\eta, b^\dagger_{q\eta'}] = 0 \quad (1)$$

Klein factors of same species: $\eta = \eta'$

$$F^\dagger_\eta F_\eta = 1 = F_\eta F^\dagger_\eta, \qquad F^\dagger_\eta F^\dagger_\eta \neq 0, \neq 1 \qquad F_\eta F_\eta \neq 0, \neq 1 \quad (2)$$

(so F is unitary)

so F is NOT a fermionic operator and also NOT a Majorana

Klein factors of different species, ANTIcommute! $\eta \neq \eta'$

$$F_\eta F_{\eta'} = - F_{\eta'} F_\eta, \qquad F^\dagger_\eta F^\dagger_{\eta'} = - F^\dagger_{\eta'} F^\dagger_\eta$$
$$F^\dagger_\eta F_{\eta'} = - F_{\eta'} F^\dagger_\eta, \qquad F_\eta F^\dagger_{\eta'} = - F^\dagger_{\eta'} F_\eta \quad (3)$$

(because the states produced by left- or right-hand side differ by interchanging two fermions)

Compact form for (2,3):

$$\{F_\eta, F^\dagger_{\eta'}\} = 2\delta_{\eta\eta'}, \qquad \text{for all } \eta, \eta' \quad (4)$$
$$\{F_\eta, F_{\eta'}\} = \{F^\dagger_\eta, F^\dagger_{\eta'}\} = 0, \qquad \text{for all } \eta \neq \eta' \quad (5)$$

ladder properties:

$$N_\eta F^\dagger_{\eta'} = F^\dagger_{\eta'}(N_\eta + \delta_{\eta\eta'}) \Rightarrow [N_\eta, F^\dagger_{\eta'}] = + \delta_{\eta\eta'} F^\dagger_\eta \quad (6)$$
$$N_\eta F_{\eta'} = F_{\eta'}(N_\eta - \delta_{\eta\eta'}) \Rightarrow [N_\eta, F_{\eta'}] = - \delta_{\eta\eta'} F_\eta \quad (7)$$

II.3 $\psi_\eta(x)|\vec{N}\rangle_0$ is an eigenstate of boson annihilation operator b (1)

Recall:

$$b_{q\eta} \overset{(2.2)}{=} \frac{-i}{\sqrt{n_q}} \sum_k c^\dagger_{k-q,\eta} c_{k\eta}, \qquad \psi_\eta(x) \overset{(1.2)}{=} \Delta_L^{1/2} \sum_{k'}{}^{'''} e^{-ik'x} c_{k'\eta} \quad (2)$$

(take strictly a=0)

Consider:

$$[b_{q\eta'}, \psi_\eta(x)] = \frac{-i}{\sqrt{n_q}} \sum_k \Delta_L^{1/2} \sum_{k'} \underbrace{[c^\dagger_{k-q,\eta'} c_{k\eta'}, c_{k'\eta}]}_{-\delta_{\eta\eta'}\delta_{k-q,k'} c_{k\eta'}} e^{-ik'x} \qquad k' = k-q \quad (3)$$

$$= \delta_{\eta\eta'} \underbrace{\left[\frac{i}{\sqrt{n_q}} e^{iqx}\right]}_{=\alpha_q(x)} \underbrace{\Delta_L^{1/2} \sum_k e^{-ikx} c_{k\eta}}_{\psi_\eta(x)} = \delta_{\eta\eta'} \alpha_q(x) \psi_\eta(x) \quad (4)$$

[Similarly: (for future reference)

$$[b^\dagger_{q\eta'}, \psi_\eta(x)] = \delta_{\eta\eta'} \alpha^*_q(x) \psi_\eta(x), \quad (5)$$
]

Act with (3) on $\vec{N}$-particle ground state:

$$b_{q\eta'} \psi_\eta(x) |\vec{N}\rangle_0 - \psi_\eta(x) \cancel{\underbrace{b_{q\eta} |\vec{N}\rangle_0}_{(4.4)\ =\ 0}} \overset{(4)}{=} \delta_{\eta\eta'} \alpha_q(x) \psi_\eta(x) |\vec{N}\rangle_0 \quad (6)$$

So, $\psi_\eta(x)|\vec{N}\rangle_0$ is an eigenstate of $b_{q\eta}$ with eigenvalue $\alpha_q(x)$ $\Rightarrow$ (1) $\square$

Properties of eigenstates of boson annihilation operators ("boson coherent states")

Consider eigenstate of b: ("Boson coherent state")

$$\boxed{b|\alpha\rangle = \alpha|\alpha\rangle \quad , \text{ with } \alpha \in \mathbb{C}} \qquad (1)$$

General form: (unnormalized)

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \qquad (\text{where } b^\dagger b|n\rangle = n|n\rangle) \qquad (2)$$

Check:

$$b|\alpha\rangle = \alpha \sum_{\substack{\cancel{n=0} \\ n=1}}^{\infty} \underbrace{\frac{\alpha^{n-1}}{\cancel{\sqrt{n!}}}}_{\frac{1}{\sqrt{(n-1)!}}} \overbrace{b|n\rangle}^{\cancel{\sqrt{n}}|n-1\rangle} \underset{n=\bar{n}+1}{=} \alpha \underbrace{\sum_{\bar{n}=0}^{\infty} \frac{\alpha^{\bar{n}}}{\sqrt{\bar{n}!}} |\bar{n}\rangle}_{|\alpha\rangle} = \alpha|\alpha\rangle \qquad (3)$$

Alternative representation: (unnormalized)

$$|\alpha\rangle = e^{\alpha b^\dagger} |0\rangle \qquad (4)$$

Check:

$$b|\alpha\rangle = \underbrace{b\, e^{\alpha b^\dagger}}_{(10.2i)} |0\rangle = e^{\alpha b^\dagger} (\underbrace{b + \alpha) |0\rangle}_{=0} = \alpha|\alpha\rangle \qquad (5)$$

$$(10.2i): \quad A e^B = e^B (A + c), \text{ with } c = [A,B] = \alpha \qquad \text{here: } A = b, \quad B = \alpha b^\dagger \qquad (6)$$

II. 4 Standard operator identities

(see vDS, App. C, for simple proofs)

1. Baker-Hausdorff:

$$e^{-B} A e^B = A + [A,B] + \ldots + \frac{1}{n!}\underbrace{[[[A,B],B] \ldots B]}_{\text{n commutators}} + \ldots \qquad (1)$$

2. [A,B] = c commutes with A and B

If $[A,B] = c$ and $[A,c] = [B,c] = 0$, then

from Baker-Hausdorff

$$(i) \quad e^{-B} A e^B = A + c \quad \text{or} \quad [A, e^B] = c\, e^B \qquad (2i)$$

generalization of (i) [Taylor expand f(A), obtain A^n by induction from (i)]

$$(ii) \quad e^{-B} f(A) e^B = f(A + c) \qquad \text{(exp(B) shifts A by c units)} \qquad (2ii)$$

(ii) with f(A) = exp(A)

$$(iii) \quad e^A e^B = e^{A+B+c/2} = e^B e^A e^c \qquad (2iii)$$

3. [A,B] = D is proportional to B

If $[A,B] = dB$ and $[A,d] = [B,d] = 0$, then

$$(i) \quad f(A) B = B f(A+d) \qquad \text{(B raises A by d units)} \qquad (3i)$$

(i) with f(A) = exp(A)

$$(ii) \quad e^A B = B e^{A+d} \qquad (3ii)$$

[Taylor expand exp(B), obtain B^n by induction from (ii)]

$$(iii) \quad e^A e^B = e^{B e^d} e^A \qquad (3iii)$$

II. V Action of fermion field on N-particle ground state

$$b_{q\eta'}\,\psi_\eta(x)\,|\vec{N}\rangle_0 \overset{(8.6)}{=} \delta_{\eta\eta'}\,\alpha_q(x)\,\psi_\eta(x)\,|\vec{N}\rangle_0 \qquad (1)$$

Coherent-state representation:

$$\psi_\eta(x)|\vec{N}\rangle_0 = \underbrace{e^{\left[\sum_{q>0}\alpha_q(x)\,b^\dagger_{q\eta}\right]}}_{\text{ensures (1)}} \underbrace{F_\eta}_{\text{lowers N}} \underbrace{\hat\lambda_\eta(x)}_{\text{prefactor (to be fixed)}}|\vec{N}\rangle_0 \overset{(3)}{=} e^{-i\varphi^\dagger_\eta(x)}\,F_\eta\,\hat\lambda_\eta(x)\,|\vec{N}\rangle_0 \qquad (2)$$

$[\ ]\ = -i\varphi^\dagger_\eta(x)$

$\sim \sum_k c_k$ lowers N, creates one p-h excitation

creates many p-h excitations

Recall (I.16.2): $$-i\,\varphi^\dagger_\eta(x) \overset{(2.7)^\dagger}{=} +\sum_{q>0} e^{-\cancel{aq/2}}\,\underbrace{\frac{i}{\sqrt{n_q}}\,e^{+iqx}}_{\alpha_q(x)}\,b^\dagger_{q\eta} \overset{(a=0)}{=} [\ ] \text{ in (2)} \qquad (3)$$

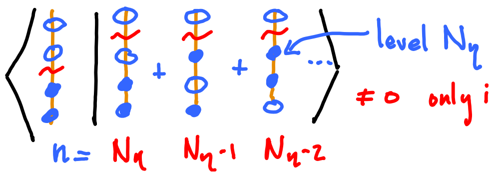
Fix prefactor: ${}_0\langle\vec{N}-1_\eta|$ (2):

$k = \Delta_L(n-1/2)$

$$\underbrace{{}_0\langle\vec{N}-1_\eta|\,\Delta_L^{1/2}\sum_{n\le N_\eta} e^{-ikx}\,c_{k\eta}|\vec{N}\rangle_0}_{\Delta_L^{1/2}\,e^{-i\Delta_L(N_\eta-1)x}} \overset{(2)}{=} \underbrace{{}_0\langle\vec{N}-1_\eta|\,e^{[\ldots b^\dagger]}}_{{}_0\langle\vec{N}|}\,F_\eta\,\hat\lambda_\eta(x)\,|\vec{N}\rangle_0 \qquad (4)$$

$e^{[\ldots b^\dagger]}$: $=1$ since ${}_0\langle\vec{N}-1_\eta|\,b^\dagger = 0$

$$// \qquad \Delta_L^{1/2}\,e^{-i\Delta_L(N_\eta-1)x} = {}_0\langle\vec{N}|\ \hat\lambda_\eta(x)\,|\vec{N}\rangle_0 \qquad (5)$$



$\neq 0$ only if $n = N_\eta$

$$\boxed{\hat\lambda_\eta(x) \overset{(5)}{=} \Delta_L^{1/2}\,e^{-i\Delta_L(\hat N_\eta-\frac{1}{2})x}} \qquad (6)$$

II.6 Action of fermion field on general N-particle state

$$|\vec{N}\rangle \underset{(1.2)}{=} f(\{b^\dagger_{q\eta}\})\,|\vec{N}\rangle_0$$

Operator identities:

$$\psi_\eta(x)\,f(\{b^\dagger_{q\eta}\}) = \underbrace{f(\{b^\dagger_{q\eta} - \delta_{\eta\eta'}\,\alpha^*_q(x)\})}_{e^{-i\varphi_\eta(x)}\,f(\{b^\dagger_{q\eta}\})\,e^{i\varphi_\eta(x)}}\,\psi_\eta(x) \qquad \left[\begin{array}{l}(10.3i),\ A = b^\dagger-\alpha,\ B=\psi \\ B\,f(A+\alpha) = f(A)\,B\end{array}\right] \quad (1)$$

$$\left[\begin{array}{l}(10.2ii),\ A = b^\dagger,\ B = i\varphi,\ C = -\alpha \\ f(A+C) = e^{-B}\,f(A)\,e^{+B}\end{array}\right] \quad (2)$$

Consider:

$$\psi_\eta(x)\,|\vec{N}\rangle = \psi_\eta(x)\,f(\{b^\dagger_{q\eta}\})\,|\vec{N}\rangle_0 \qquad (3)$$

$$\overset{(1)}{=} f(\{b^\dagger_{q\eta} - \delta_{\eta\eta'}\,\alpha^*_q(x)\})\,\underbrace{\psi_\eta(x)\,|\vec{N}\rangle_0}_{\overset{(11.2)}{=}\ e^{-i\varphi^\dagger_\eta(x)}\,F_\eta\,\hat\lambda_\eta(x)\,|\vec{N}\rangle_0} \qquad (4)$$

since

$[b^\dagger, \varphi^\dagger] = [b^\dagger, F] = [b^\dagger, N] = 0$

since

$$= e^{-i\varphi^\dagger_\eta(x)}\,F_\eta\,\hat\lambda_\eta(x)\,\overset{(2)}{e^{-i\varphi_\eta(x)}}\,\underbrace{f(\{b^\dagger_{q\eta}\})\,e^{i\varphi_\eta(x)}\,|\vec{N}\rangle_0}_{|\vec{N}\rangle} \qquad (5)$$

$e^{i\varphi_\eta(x)}$: $=1$, since $b|\vec{N}\rangle_0 = 0$

$$\boxed{\psi_\eta(x)\,|\vec{N}\rangle = F_\eta\,\hat\lambda_\eta(x)\,e^{-i\varphi^\dagger_\eta(x)}\,e^{-i\varphi_\eta(x)}\,|\vec{N}\rangle} \qquad (6)$$

holds for all $|\vec{N}\rangle$
hence as operator
identity for $\psi = F\lambda\,e^{-i\varphi^\dagger}e^{-i\varphi}$