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Bosonization for Beginners

Lectures given by Jan von Delft
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In 1D, "bosonization relations" of the following form hold:

$$\psi \sim F e^{-i\phi}$$

fermion field — Klein factor — boson field

Goal of lectures:
- explain origin of these relations
- illustrate them with some canonical examples

Outline:

I. 1D-fermions, 1D-bosons
II. Bosonization identity
III. Impurity in Luttinger Liquid
IV. Kondo model

Literature:
- *Bosonization for Beginners - refermionization for experts*, Jan von Delft & Herbert Schoeller, Ann. Physics 7, 225-306 (1998), cond-mat/9805275
- *Simple Bosonization Solution of the 2-channel Kondo Model: I. Analytical Calculation of Finite-Size Crossover Spectrum*, Gergely Zarand and Jan von Delft, Phys. Rev. B. **61**, 6918 (2000) [including appendices: cond-mat/9812192]
- *Interacting fermions in one dimension: The Tomonaga-Luttinger model* K. Schönhammer, cond-mat/9710330

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Popular applications

(pioneered by: Luttinger, Schotte & Schotte, Mattis &Lieb, Luther & Peschel, Haldane
applications: Kane & Fisher, Wen, Shankar...)

1. Interactions in 1D

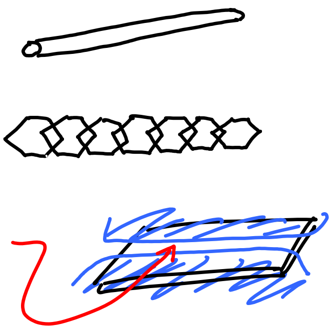
Since fermions in 1D cannot pass each other, interactions are "strong" and dramatically change the physics (e.g. spin-charge separation)

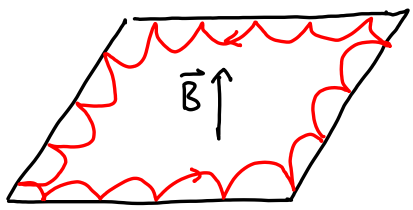
Applications:

nanotubes

organic molecules

semiconductor quantum wires





quantum Hall edge states

Interactions in 1D: $\int dx\, \psi^\dagger \psi\, \underbrace{\psi^\dagger \psi}_{(\partial_x \phi)} \sim \int dx\, (\partial_x \phi)^2$

Kinetic energy: $\int dx\, \psi^\dagger \partial_x \psi \sim \int dx\, (\partial_x \phi)^2$

QUADRATIC !

Interacting model becomes exactly solvable!

2. Impurity models (Kondo):

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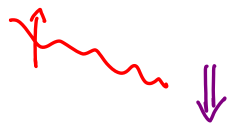
(Emery & Kivelson, '92)

$|\psi_{in}\rangle$:  $|\psi_{out}\rangle$: 

Spin-flip term: $S^+ \psi_\downarrow^\dagger(0)\, \psi_\uparrow(0) \;+\; S^- \psi_\uparrow^\dagger(0)\, \psi_\downarrow(0)$

Bosonize: $\sim S^+ e^{i(\phi_\downarrow - \phi_\uparrow)} + S^- e^{i\underbrace{(\phi_\uparrow - \phi_\downarrow)}_{\phi_s}}$

$\psi_\sigma \sim e^{-i\phi_\sigma}$

Warning: I'm being sloppy here... See lecture 4

New boson field: $\sim S^+ \underbrace{e^{-i\phi_s}}_{\psi_s} + S^- \underbrace{e^{i\phi_s}}_{\psi_s^\dagger}$

Refermionize: $d^\dagger \psi_s + d\, \psi_s^\dagger$ ← QUADRATIC !!

Heuristic plausibility argument for bosonization relation

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How can it be true that: $\psi \sim e^{-i\phi}$? (1)

For 1-D bosons, with linear dispersion: $\langle \phi(x)\, \phi(0) \rangle \sim -\ln x$ (2)

For 1-D fermions, with linear dispersion: $\langle \psi^\dagger(x)\, \psi(0) \rangle \sim \frac{1}{x}$ (3)

or, using (1): $\sim \left\langle e^{i\phi(x)}\, e^{-i\phi(0)} \right\rangle$ (4)

standard identity for bosonic operators: $\sim e^{\langle \phi(x)\phi(0) - \phi(0)\phi(0)\rangle}$ (5)

using (2): $\sim e^{-\ln x} \sim \frac{1}{x} = (3)\,\checkmark$ (6)

Questions:

$$\psi_\sigma \sim F_\sigma e^{-i\phi_\sigma} \qquad (\sigma = \uparrow, \downarrow) \qquad (1)$$

$\boxed{5}$

How general is (5.1) ? only in 1D , infinite bandwidth

Does (5.1) rely on linear dispersion? NO !

Is (5.1) an operator identity? YES ! On what Fock space ?

Commutation relations? $[\phi(x), \partial_x \phi(0)] = \delta(x) \overset{!!}{\Longleftrightarrow} \{\psi(x), \psi^\dagger(0)\} = \delta(x)$

Several species of electrons? $\{\psi_\uparrow(x), \psi^\dagger_\downarrow(0)\} \overset{?}{=} 0 \Longleftrightarrow \{F_\downarrow, F^\dagger_\uparrow\} = 0$

Klein factors!

Role of cut-offs ? Infrared: $\frac{1}{L}$ Ultraviolet: $\Lambda \sim \frac{1}{a}$

Finite-size effects? $\frac{1}{L} \neq 0$ Useful !!

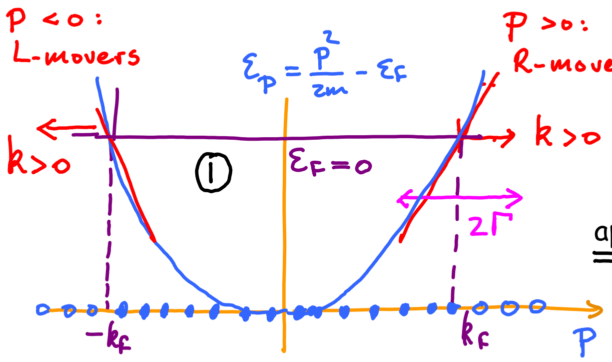
Outline of lecture I: 1-D fermions & bosons

1. Linearization of fermion spectrum
2. Properties of 1d fermion fields
3. 1D fermion correlators
4. Normal ordering
5. Density fluctuations - bosonic excitations
6. Properties of 1d boso

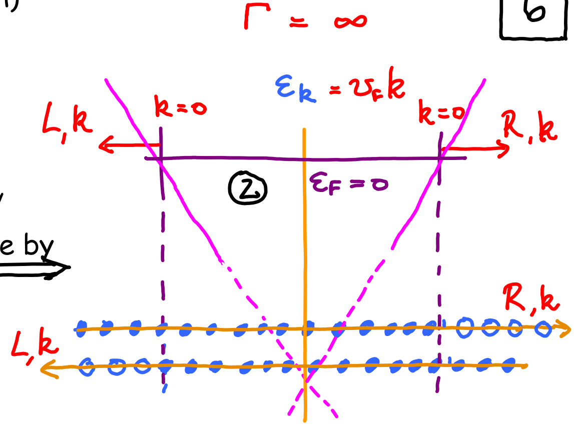
I.1 Linearization of fermion spectrum

(ignore spin)

$\boxed{6}$



replace/ approximate by



$k = |p| - k_F$, or $p = \pm(k_F + k)$ for R/L

add positron states with $k \to -\infty$ for L/R-branches

① For $|k| \ll \Gamma \simeq p_F$, linearization is justified:

$$\varepsilon_p = \frac{|p|^2 - p_F^2}{2m} = \frac{\cancel{p_F^2} + 2p_F k + k^2 - \cancel{p_F^2}}{2m} = v_F k \left(1 + \frac{k}{2p_F}\right) \qquad (1)$$

$$\simeq v_F k \quad \text{if} \quad |k| < \Gamma \simeq p_F$$

curvature-effect

(2)

Neglected terms [order (k/kF)] describe curvature effects: current research topic!
Fermi-Luttinger liquid: Spectral function of interacting one-dimensional fermions, Khodas, Pustilnik, Kamenev, Glazman, PRB, 76, 155402 (2007)

Replacing (6.1) by (6.2) is justified if we are interested only in long-wavelength / low/energy **7**

properties, with $|q| \ll \Gamma$ anyway, i.e. in excitation energies ω, T, V

In this case, we may as well send cutoff $\Gamma \to \infty$, and replace theory ① → ②

Corresponding approximation for electron fields, step by step:

write $p = \pm(k_F + k)$

$$\psi_{phys}(x) = \Delta_L^{1/2}\sum_p e^{ipx} c_p = \Delta_L^{1/2}\sum_{k=-k_F}^{\infty}\Big(e^{-i(p_F+k)x}\underbrace{c_{-p_F-k}}_{c_{k,L}} + e^{+i(p_F+k)x}\underbrace{c_{+p_F+k}}_{c_{k,R}}\Big) \qquad (1)$$

$\Delta_L = \left(\frac{2\pi}{L}\right)$

$$\hookrightarrow \simeq \Big[\underbrace{\sum_{|k|<\Gamma}}_{Ⓐ} + \underbrace{\sum_{|k|>\Gamma \ \text{and}\ k>-k_F}}_{Ⓑ}\Big]$$

Drop high-energy excitations, assuming they
don't matter for low-energy properties:

Step 1: drop B

$$\longrightarrow \psi_{phys}(x) \text{ "=" } e^{-ip_Fx}\psi_L(x) + e^{ip_Fx}\psi_R(-x) \qquad (2)$$

with

$$\psi_{L/R}(x) := \Delta_L^{1/2}\sum_{|k|<\Gamma} e^{-ikx} c_{k,L/R} \qquad (3)$$

I.2 Properties of 1d fermion fields

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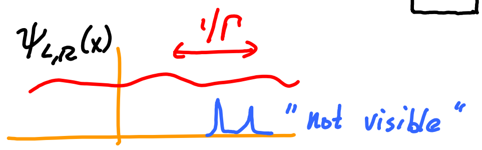
Cutoff means: new fields $\psi_{L/R}(x)$ can resolve
spatial structures only if they are coarser than $1/\Gamma$;



Step 2: to get a mathematically simpler, cleaner theory,
now take cutoff to infinity, i.e. add "positron states"
(since they did not matter for low excitation energies anyway):

$$\sum_{|k|<\Gamma \simeq a} \longrightarrow \underbrace{\lim_{a\to 0}\sum_{k=-\infty}^{\infty} e^{-|k|a}}_{\sum'_k}$$

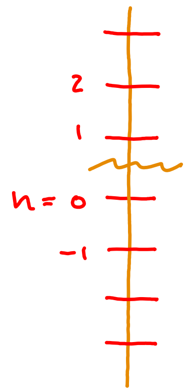
(implicit)

So, write:
$\eta = L, R$

$$\boxed{\text{(x is smeared on scale a)} \quad \psi_\eta(x) = \Delta^{1/2}\sum_k{}' e^{-ikx} c_{k\eta}} \qquad (1)$$

Impose anti-periodic
boundary conditions:
(convenient to avoid degeneracy
of Fermi ground state)

$$\psi_\eta(-L/2) = \psi_\eta(L/2) \Rightarrow \boxed{k = \underbrace{\frac{2\pi}{L}}_{\Delta_L}\left(n - \frac{1}{2}\right)} \qquad (2)$$



Anticommutators: $\{c_{k\eta}, c_{k'\eta'}\} = 0, \quad \{c_{k\eta}, c^\dagger_{k'\eta'}\} = \delta_{kk'}\delta_{\eta\eta'}$ (1) $\boxed{9}$

$$\{\psi_\eta(x), \psi_{\eta'}(x')\} = 0 \qquad (2)$$

$$\{\psi_\eta(x), \psi^\dagger_{\eta'}(x')\} \overset{(8.1)}{=} \Delta_L \sum_{kk'}{}' \sum_{\eta\eta'} e^{-ixk} e^{+ix'k'} \overbrace{\{c_{k\eta}, c^\dagger_{k'\eta'}\}}^{\delta_{kk'}\delta_{\eta\eta'}} \qquad (3)$$

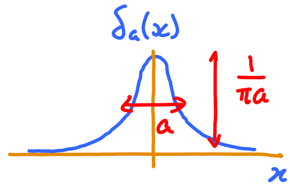
$$= \delta_{\eta\eta'} \underbrace{\Delta_L \sum_k}_{\int dk} e^{-ik(x-x')} e^{-|k|a} \qquad (4)$$

Continuum limit:
(finite bandwidth) $\xrightarrow{L\to\infty}$

$$= \delta_{\eta\eta'}\, 2\pi \underbrace{\frac{a/\pi}{(x-x')^2 + a^2}} \qquad (5)$$

convention of vDS
(usually 1)

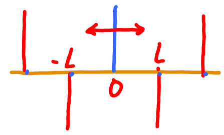
$$= \boxed{\delta_{\eta\eta'}\, 2\pi\, \delta_a(x-x')}$$



smeared delta-function

Or:
infinite bandwidth $\{\psi_\eta(x), \psi^\dagger_{\eta'}(x')\} = \delta_{\eta\eta'} \Delta_L \sum_{n\in\mathbb{Z}} e^{-i(n-1/2)(x-x')\Delta_L}$ (6)

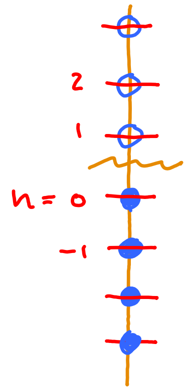
$$= \delta_{\eta\eta'}\, 2\pi \sum_{\bar n} \delta(x - x' - \bar n L)\,(-1)^{\bar n}$$



antiperiodic delta-function

Linearized kinetic energy: $H \overset{(6.2)}{=} \sum_{k\eta}{}' v_F k\, c^\dagger_{k\eta} c_{k\eta}$ (1) $\boxed{10}$

Fermi ground state:
$$\begin{cases} k < 0 \quad \text{filled}: & c^\dagger_{k\eta}|0\rangle = 0 \\ k > 0 \quad \text{empty}: & c_{k\eta}|0\rangle = 0 \end{cases} \qquad (2)$$



Imaginary-time evolution: $c_{k\eta}(\tau) := e^{H\tau/\hbar} c_{k\eta} e^{-H\tau/\hbar}$ (3)

$$= e^{-\underbrace{(v_F/\hbar)}_{\equiv 1} k\tau} c_{k\eta} = e^{-k\tau} c_{k\eta} \qquad (4)$$

[If we ever need real-time evolution: $c_k(t) = c_k(\tau \to it) = e^{-ikt} c_k$ ✓ (5)]

Fermion field: $\psi_\eta(\tau, x) = \Delta_L^{1/2} \sum_k{}' e^{-k\underbrace{(ix+\tau)}_{=:z}} c_{k\eta} = \Delta_L^{1/2} \sum_k e^{-kz} c_{k\eta} = \psi(z)$ (6)

I.3 Imaginary-time-ordered fermion correlator at T = 0

$$\psi_\eta(z) \overset{(10.6)}{=} \Delta_L^{1/2} \sum_k{}' e^{-kz} c_k$$

$$-\mathcal{G}_{\eta\eta'}(z) = \langle \mathcal{T}\, \psi_\eta(z)\, \psi^\dagger_{\eta'}(0) \rangle \qquad (1)$$

$$= \theta(\tau)\langle \psi_\eta(z)\,\psi^\dagger_{\eta'}(0)\rangle - \theta(-\tau)\langle \psi^\dagger_{\eta'}(0)\,\psi_\eta(\tau)\rangle \qquad (2)$$

$$= \Delta_L \sum_{kk'}{}' e^{-kz}\Big[\theta(\tau)\langle 0|\underbrace{c_{k\eta}c^\dagger_{k'\eta'}}_{\delta_{kk'}\delta_{\eta\eta'}\theta(k')}|0\rangle - \theta(-\tau)\langle 0|\underbrace{c^\dagger_{k'\eta'}c_{k\eta}}_{\delta_{k'k}\delta_{\eta'\eta}\theta(-k)}|0\rangle\Big] \qquad (3)$$

$$= \delta_{\eta\eta'}\ \underbrace{\Delta_L \sum_{k>0}^{\infty}}_{\int_0^\infty dk} e^{-kz\sigma}\, e^{-ka}\cdot\sigma \qquad [\sigma = \mathrm{sign}(\tau)] \qquad (4)$$

$$\xrightarrow{L\to\infty} \delta_{\eta\eta'}\,\sigma\left[\frac{e^{-z\sigma-a}}{-z\sigma-a}\right]_0^\infty = \boxed{\delta_{\eta\eta'}\frac{1}{z+\sigma a}} \qquad (5)$$

a regularizes the correlator for z = 0

For finite L one finds, using $k = \Delta_L(n+1/2)$, $y := e^{-\Delta_L(\sigma z + a)}$:

$$-\mathcal{G}_{\eta\eta'}(z) = \Delta_L\,\sigma\, y^{-\sigma/2}\sum_{n=0}^{\infty} y^n = \Delta_L\,\sigma\,\frac{y^{-(\sigma+1)/2}}{y^{-1/2}-y^{1/2}} = \frac{\delta_{\eta\eta'}\, e^{\pi(\sigma+1)/L}}{\frac{L}{\pi}\sinh\left[\frac{\pi}{L}(z+\sigma a)\right]} \qquad (6)$$

I.4 Fermion normal ordering

$A, B, C \in \{c_{k\eta}, c^\dagger_{k'\eta'}\}$

$$\left\{\begin{array}{ll} c_{k\eta}, & \text{for } k>0 \\ c^\dagger_{k\eta}, & \text{for } k<0 \end{array}\right\} \qquad \left\{\begin{array}{ll} c_{k\eta}, & \text{for } k<0 \\ c^\dagger_{k\eta}, & \text{for } k>0 \end{array}\right\} \qquad (1)$$

To bring "normal order" a product of operators,
move all operators that annihilate the vacuum to the right of all others,
and multiply by (-1) for each exchange of two fermion operators. (2)

For product of two operators, this is equivalent to:

$${}^\times_\times AB\,{}^\times_\times = AB - \langle 0|AB|0\rangle \qquad (3)$$

Example: $k<0,\ k'<0$:

$${}^\times_\times c^\dagger_k c_{k'}\,{}^\times_\times = -c_{k'}c^\dagger_k \qquad (4)$$

$$\overset{(9.1)}{=} c^\dagger_k c_{k'} - \underbrace{\delta_{kk'}}_{\langle c^\dagger_k c_{k'}\rangle} \overset{\checkmark}{=} (1) \qquad (5)$$

By definition, vacuum expectation value of two normal ordered operators vanishes:

$$\langle 0|{}^\times_\times AB\,{}^\times_\times|0\rangle \overset{(3)}{=} 0 \qquad (6)$$

<u>I.5 Density fluctuations - bosonic excitations</u>

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(2 pi) density: $$\rho_\eta(x) = :\psi^\dagger_\eta(x)\,\psi_\eta(x): \qquad (1)$$

bilenear, hence bosonic in character!

$$= \Delta_L \sum_{\cancel{k}q}{}' e^{i(\cancel{k}-q-\cancel{k})x} \overbrace{\sum_k :c^\dagger_{k-q\eta}\, c_{k\eta}:} \qquad (2)$$

Fourier representation:
i.t.o. density modes:

$$= \Delta_L \hat{N}_\eta + \Delta_L \sum_{q>0} i\sqrt{n_q}\left(e^{-iqx} b_{q\eta} - e^{iqx} b^\dagger_{q\eta}\right) \qquad (3)$$

$$\left[q = \Delta_L n_q \, , \; n_q \in \mathbb{Z}\right]$$

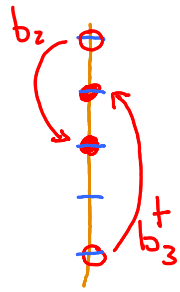
where we defined:

Particle number relative to Fermi ground state:
$$\hat{N}_\eta = \sum_k :c^\dagger_{k\eta} c_{k\eta}: \qquad (4)$$
[the q = 0 term of (2)]

Momentum lowering op:
(Bosonic annihilation op)
$$b_{q\eta} = \frac{-i}{\sqrt{n_q}} \sum_k c^\dagger_{k-q,\eta}\, c_{k\eta} \qquad (5)$$

$\boxed{q>0}$

Momentum raising op:
(Bosonic creation op)
$$b^\dagger_{q\eta} = \frac{i}{\sqrt{n_q}} \sum_k c^\dagger_{k+q,\eta}\, c_{k\eta} \qquad (6)$$



Note: (5) and (6) are automatically normal ordered, hence no need to write : :

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$$[A, BC]_- = [A,B]_\mp C \pm B[A,C]_\mp \qquad (1)$$

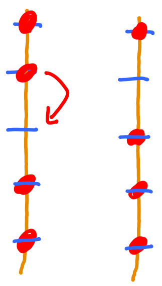
$$[a^\dagger b, c^\dagger d]_- = [a^\dagger b, c^\dagger]_- d + c^\dagger [a^\dagger b, d]_- = a^\dagger d\, \delta_{bc} - c^\dagger b\, \delta_{ad} \qquad (2)$$

Bosonic commutation relations: (for notational simplicity, below we drop the index η)

$$[N, b_q] = \frac{-i}{\sqrt{n_q}} \sum_{kk'} \left[c^\dagger_k c_k \, , c^\dagger_{k'-q} c_{k'}\right] \qquad (3)$$

$$\overset{(2)}{=} \frac{-i}{\sqrt{n_q}} \sum_{kk'} \left(c^\dagger_k c_{k'}\, \delta_{k,k'-q} - c^\dagger_{k'-q} c_k\, \delta_{kk'}\right) \qquad (4)$$

cancel



momentum-lowering operator does not change particle number

$$= \frac{-i}{\sqrt{n_q}} \sum_k \left(c^\dagger_k c_{k+q} - c^\dagger_{k-q} c_k\right) = 0 \qquad (5)$$

shift: $k \to k+q$

Similarly: $$[N, b^\dagger_q] = 0 \, , \quad [b_q, b_{q'}] = 0 \, , \quad [b^\dagger_q , b^\dagger_{q'}] = 0 \qquad (6)$$

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$$\boxed{\delta_{qq'} = [b_q, b^\dagger_{q'}]} = \frac{1}{n_q}\sum_{kk'}\left[c^\dagger_{k-q}\, c_k, c^\dagger_{k'+q'}\, c_{k'}\right] \quad (1)$$

$[a^\dagger b, c^\dagger d] = a^\dagger d\, \delta_{bc} - c^\dagger b\, \delta_{ad}$

$$= \frac{1}{n_q}\sum_{kk'}\left[c^\dagger_{k-q}\, c_{k'}\, \delta_{k,k'+q'} - c^\dagger_{k'+q'}\, c_k\, \delta_{k-q,k'}\right] \quad (2)$$

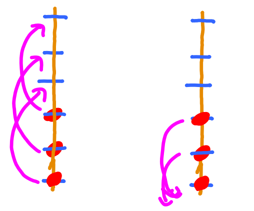
$$= \frac{1}{n_q}\sum_{k}\left[c^\dagger_{k-q}\, c_{k-q'} - c^\dagger_{\underset{k-q}{k-q+q'}}\, c_{\underset{k+q'}{k}}\right] \quad (3)$$

cancel if $q \neq q'$

if $q \neq q'$, both terms are normal-ordered, so we can set $k+q' \to k$ here, obtaining 0.
if $q = q'$, both terms have to be normal-ordered first, before rearranging sum; this gives:

number of possible transitions generated by

$b_3^\dagger$: 3 $\qquad$ b_3 : 0



$$= \delta_{qq'}\frac{1}{n_q}\sum_k\left[:c^\dagger_{k-q}\, c_{k-q}: - :c^\dagger_k\, c_k: \right. \quad (4)$$

cancel after shift in first term: $k \to k+q$

$$\left. + \langle 0|c^\dagger_{k-q}\, c_{k-q}|0\rangle - \langle 0|c^\dagger_k\, c_k|0\rangle\right] \quad (5)$$

$$= \delta_{qq'}\frac{1}{n_q}\left[\sum_{n_k=-\infty}^{n_q} - \sum_{n_k=-\infty}^{0}\right] = \frac{\delta_{qq'}}{n_q}\cdot n_q = \delta_{qq'} \quad (6)$$

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6. Properties of 1d Boson fields

$q = \Delta_L n_q$

"annihilation field": $$\boxed{\varphi_\eta(x) = -\overbrace{\sum_{q>0} e^{-aq/2}}^{=\sum'}\frac{1}{\sqrt{n_q}}\, e^{-iqx}\, b_{q\eta}} \quad (1)$$

"creation field": $$\boxed{\varphi^\dagger_\eta(x) = -\sum_{q>0} e^{-aq/2}\frac{1}{\sqrt{n_q}}\, e^{+iqx}\, b^\dagger_{q\eta}} \quad (2)$$

The ultraviolet cutoff a here acts as a bandwidth for bosonic excitations.
In fermion language, it sets the maximum momentum difference between particle/hole pairs.

Hermitian boson field: $$\boxed{\phi_\eta(x) = \varphi_\eta(x) + \varphi^\dagger_\eta(x)} \quad (3)$$

Derivative gives density:
(provided a = 0)
$$\boxed{\partial_x \phi_\eta(x) = \Delta_L\sum_q i\sqrt{n_q}\left(e^{-iqx}\, b_{q\eta} - e^{iqx}\, b^\dagger_{q\eta}\right)} \quad (4)$$

Compare (16.4) & (13.3):
$$\boxed{\rho_\eta(x) = :\psi^\dagger_\eta(x)\,\psi_\eta(x): = \Delta_L N_\eta + \partial_x \phi_\eta} \quad (5)$$

<u>Boson field commutators:</u> (for notational simplicity, below we drop the index η) **17**

$$[b_q, b_{q'}] \overset{(14.6)}{=} [b_q^\dagger, b_{q'}^\dagger] = 0 \quad \Rightarrow \quad [\varphi(x), \varphi(x')] = 0, \quad [\varphi^\dagger(x), \varphi^\dagger(x')] = 0, \qquad (1)$$

$$[b_q, b_{q'}^\dagger] = \delta_{qq'} \quad \Rightarrow \quad \boxed{[\varphi(x), \varphi^\dagger(x')]} = \sum_{qq'}{}' \frac{1}{\sqrt{n_q}} \frac{1}{\sqrt{n_{q'}}} e^{-iqx} e^{iq'x'} \underbrace{[b_q, b_{q'}^\dagger]}_{\delta_{qq'}} \qquad (2)$$

$$q = \Delta_L n$$

$$y = e^{-i\Delta_L(x - x' - ia)}$$

$$\xrightarrow{L\to\infty} 1 - i\Delta_L(x - x' - ia)$$

$$= \sum_{q>0} \frac{1}{n_q} e^{-iq(x-x')} e^{-aq} \qquad (3)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} y^n = -\ln(1-y) \qquad (4)$$

$$\xrightarrow{L\to\infty} \boxed{-\ln[i\Delta_L(x - x' - ia)]} \qquad (5)$$

Note: this commutator needs both infrared and ultraviolet regulators, 1/L and a, respectively!

<u>Commutator of phi with its derivative</u> **18**

$$\boxed{[\phi(x), \partial_{x'}\phi(x')]} = [\varphi(x), \partial_{x'}\varphi^\dagger(x')] + [\varphi^\dagger(x), \partial_{x'}\varphi(x')] \qquad (1)$$

$$[\varphi(x), \varphi^\dagger(x')] \overset{(17.4)}{=} -\ln(1-y) \quad = \quad i\Delta_L \frac{\cancel{y}}{y^{-1}\,1 - \cancel{y}} - \text{c.c} \qquad (2)$$

$$y = e^{-i\Delta_L(x - x' - ia)}$$

$$\xrightarrow{L\to\infty} i\cancel{\Delta_L}\left[\frac{1}{i\cancel{\Delta_L}(x - x' - ia)} - \frac{1}{2}\right] - \text{c.c.} \qquad (3)$$

(but retain 1/L term)

$$\frac{1}{e^\Delta - 1}$$

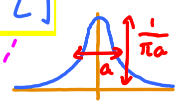
$$= \frac{1}{(1 - \Delta + \Delta^2/2 + \dots) - 1}$$

$$= \frac{1}{\Delta(1 + \Delta/2)}$$

$$= \frac{1}{\Delta}(1 - \Delta/2 + \dots)$$

$$= \frac{1}{\Delta} - \frac{1}{2}$$

$$= 2\pi i \left[\frac{a/\pi}{(x - x')^2 + a^2} - \frac{1}{L}\right] = \boxed{2\pi i\left[\delta_a(x) - \frac{1}{L}\right]} \qquad (4)$$



The 1/L term ensures consistency upon integrating (1):

$$\int_{-L/2}^{L/2} dx' \, [\phi(x), \partial_{x'}\phi(x')] \overset{(4)}{=} 2\pi i \,(1 - 1) = 0 \qquad (19)$$

$$= [\phi(x), \underbrace{\phi(L/2) - \phi(-L/2)}_{=0}] \qquad (20)$$

since phi is periodic, (16.1, 16.2)

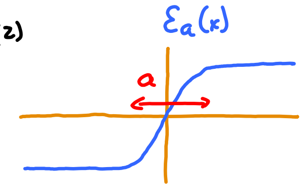
consistent!

<u>Commutator of phi with itself</u> [can be obtained by integrating (18.1)]

$$\boxed{[\phi(x), \phi(x')]} = \int^{x'} d\bar{x}\, [\phi(x), \partial_{\bar{x}} \phi(\bar{x})] + C \qquad (1)$$

fixed by requiring commutator to vanish for x = x'

$$\overset{(18.4)}{=} 2i \int^{x'} d\bar{x} \left[\frac{a}{(x-\bar{x})^2 + a^2} - \frac{\pi}{L} \right] \qquad (2)$$



smeared step function

$$= -2i \left[\arctan\left(\frac{x-x'}{a}\right) - \frac{\pi(x-x')}{L} \right] \qquad (3)$$

$$= -i\pi\, \mathcal{E}_a(x-x') \qquad (4)$$

where

$$\boxed{\mathcal{E}_a(x) = \begin{cases} \pm 1 & \text{for } x \gtrless 0 \\ 0 & \text{for } x = 0 \end{cases}} \qquad (5)$$

(4) is the form most often quoted, with a = 0, L = infinity.