TMP-TC2: Cosmology

Solutions to Problem Set 11

4 July 2023

1. Equations of motion for the inflaton

We have a theory described by the following action

$$S = S_g + S[\phi] \ ,$$

where the gravitational part S_g is the usual Einstein-Hilbert action

$$S_g = \int d^4x \frac{1}{16\pi G} R \ ,$$

and the inflaton's part is

$$S[\phi] = \int d^4x \sqrt{-g}\ \mathcal{L}[\phi] = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \ .$$

The variation of the action with respect to ϕ yields

$$\begin{aligned} \delta_\phi S[\phi] &= \int d^4x \sqrt{-g} \left(-g^{\mu\nu} \partial_\nu \phi \partial_\mu \delta\phi - V'(\phi)\delta\phi \right) \\ &= \int d^4x \left[\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) - \sqrt{-g} V'(\phi) \right] \delta\phi \ , \end{aligned}$$

so the equation of motion for the field is

$$\frac{1}{\sqrt{-g}} \partial_\mu \left[\sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right] - V'(\phi) = 0 \ .$$

We now introduce to the above the explicit form of the (spatially flat) FLRW metric

$$ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2) \quad \rightarrow \quad g_{\mu\nu} = \mathrm{diag}[-1, a(t)^2, a(t)^2, a(t)^2] \ ,$$

and use $g = -a(t)^6$, therefore

$$\frac{1}{a(t)^3} \partial_\mu \left[a(t)^3 g^{\mu\nu} \partial_\nu \phi \right] - V'(\phi) = 0 \quad \rightarrow \quad \frac{1}{a(t)^3} \partial_0 \left[a(t)^3 g^{00} \partial_0 \phi \right] + g^{ii} \partial_i \partial_i \phi - V'(\phi) = 0 \ .$$

If we take into account that the inflaton is homogeneous, the spatial derivatives can be neglected so the above gives us

$$\ddot{\phi} + 3H\phi + V'(\phi) = 0 \ ,$$

where $H = \dot{a}/a$ is the Hubble parameter.
The second equation comes from the variation of the action with respect to the metric. We have seen that the variation of the Einstein-Hilbert action S_g with respect to the metric yields the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - 1/2 g_{\mu\nu} R$. The variation of the action for the scalar field $S[\phi]$ with respect to the metric is

$$\delta_g S_\phi = \int d^4x \left(-\frac{1}{2}\sqrt{-g} g_{\mu\nu} \ \mathcal{L}[\phi] - \frac{1}{2}\sqrt{-g}\partial_\mu\phi\partial_\nu\phi \right) \delta g^{\mu\nu} \ .$$

Identifying the energy-momentum tensor with

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S[\phi]}{\delta g^{\mu\nu}} \ ,$$

we get

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L}[\phi] \ .$$

Putting everything together we get Einstein's equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \ .$$

The above for the 00 component in the FLRW space (look also at the Problem Set 1) is

$$G_{00} = 8\pi G T_{00} \quad \rightarrow \quad 3H^2 = 8\pi G \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) ,$$

where we neglected again the spatial variations of ϕ.

2. Scalar field in FLRW spacetime

1. We have seen that the energy momentum tensor $T_{\mu\nu}$ of a scalar field ϕ reads

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}\left(\partial_\kappa\phi\right)^2 - V(\phi) \right]$$

where $V(\phi)$ is the potential.
If you want, you can compare it with the one of a perfect fluid :

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - g_{\mu\nu} p \ ,$$

by doing the following identifications

$$\begin{cases} u^\mu = \dfrac{\partial^\mu \phi}{\sqrt{\partial_\nu \phi \partial^\nu \phi}} \\ p = \dfrac{1}{2}\partial^\mu \phi \partial_\mu \phi - V(\phi) \\ \rho = \dfrac{1}{2}\partial^\mu \phi \partial_\mu \phi + V(\phi) \end{cases}$$

In the given frame, the energy density of the field corresponds to

$$\rho = T_{00} = \frac{\dot{\phi}^2}{2} + \frac{1}{2a^2}\left(\partial_i \phi\right)^2 + V(\phi)$$

where dot denotes derivative with respect to time and a is the scale factor. The pressure p is related to the spatial components of the energy-momentum tensor as

$$p = \frac{1}{3}T^i_i = \frac{\dot{\phi}^2}{2} - \frac{1}{6a^2}\left(\partial_i \phi\right)^2 - V(\phi)$$

Using the above, we see that the equation of state parameter is

$$w \equiv \frac{p}{\rho} = \frac{\frac{\dot{\phi}^2}{2} - \frac{1}{6a^2}\left(\partial_i \phi\right)^2 - V(\phi)}{\frac{\dot{\phi}^2}{2} + \frac{1}{2a^2}\left(\partial_i \phi\right)^2 + V(\phi)}$$

2. In a previous exercise, we showed that accelerated expansion requires

$$p < -\frac{\rho}{3}$$

For the scalar field, the above condition gives us

$$\dot{\phi}^2 < V(\phi)$$

which means that the potential energy of the field must dominate over its kinetic energy.

3. If we assume that the field is homogeneous, i.e. $\phi \equiv \phi(t)$, the expressions for ρ and p we found in eqs. (35) and (36) simplify significantly

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi) \quad \text{and} \quad p = \frac{\dot{\phi}^2}{2} - V(\phi)$$

Using the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0$$

we immediately find

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$

which is the Klein-Gordon equation of a (homogeneous) scalar field in a flat FLRW universe.

3. Behaviour of the inflaton

The equations of motion for a free massive scalar field are

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \ , \tag{1}$$

$$H^2 = \frac{8\pi G}{6}\left(\dot{\phi}^2 + m^2\phi^2\right) \ . \tag{2}$$

1. Solving the second equation for H and putting it into the first, we get

$$\ddot{\phi} + \sqrt{12\pi G}\sqrt{\dot{\phi}^2 + m^2\phi^2}\dot{\phi} + m^2\phi = 0 \ . \tag{3}$$

This is a nonlinear second order differential equation with no explicit time dependence. It can therefore be reduced to a first order differential equation for $\dot{\phi}(\phi)$. Using $\ddot{\phi} = \dot{\phi}\frac{d\dot{\phi}}{d\phi}$, we get

$$\frac{d\dot{\phi}}{d\phi} = -\frac{\sqrt{12\pi G}\sqrt{\dot{\phi}^2 + m^2\phi^2}\dot{\phi} + m^2\phi}{\dot{\phi}} \ . \tag{4}$$

2. (a) <u>"Ultra-hard" period ($\dot{\phi} \gg m\phi$ and $\dot{\phi}^2 \gg \frac{m^2}{\sqrt{12\pi G}}\phi$)</u>

This is the situation where the potential energy is small compared to the kinetic energy. In this approximation, eq. (4) becomes

$$\frac{d\dot{\phi}}{d\phi} \approx -\sqrt{12\pi G}\dot{\phi} \ , \tag{5}$$

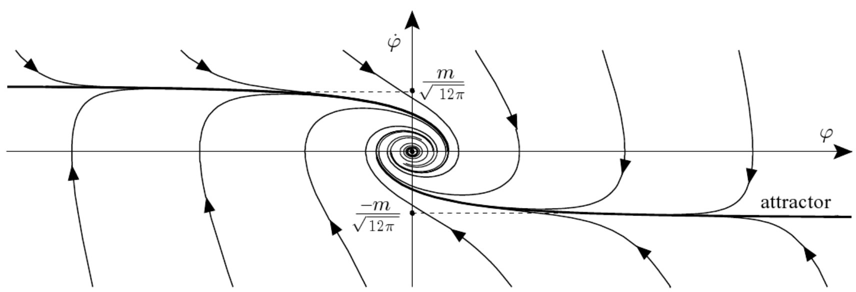
so the solution is damped exponentially

$$\dot{\phi} \approx C \exp(-\sqrt{12\pi G}\phi) \ , \tag{6}$$

where C a constant of integration. We can immediately solve this relation for $\phi(t)$ to obtain

$$\phi(t) = \frac{1}{\sqrt{12\pi G}} \log\left[C\sqrt{12\pi G}\right] + \frac{1}{\sqrt{12\pi G}} \log t \ . \tag{7}$$

The above relations tell us that even if $\dot{\phi}$ had a large initial value, it decays exponentially faster than the value of the scalar field itself. Therefore, the attractor is reached very quickly and this enlarges the set of initial conditions which lead to an inflationary stage.



In order to find the Hubble parameter, we use eq. (2), which in the "ultra-hard" limit reads

$$H^2 \approx \frac{4\pi G}{3}\dot{\phi}^2 \ . \tag{8}$$

Plugging (7) into the above, we obtain

$$H^2 \approx \frac{1}{9t^2} \quad \rightarrow \quad H \approx \frac{1}{3t} \ . \tag{9}$$

(b) <u>"Slow-roll" period ($d\dot{\phi}/d\phi \approx 0$ and $\dot{\phi}^2 \ll m^2\phi^2$)</u>

In this limit, eq. (4) gives

$$-\sqrt{12\pi G} - \frac{m}{\dot{\phi}} \approx 0 \quad \rightarrow \quad \dot{\phi} \approx -\frac{m}{\sqrt{12\pi G}} \ . \tag{10}$$

Using this result, we find that

$$\phi(t) \approx -\frac{mt}{\sqrt{12\pi G}} \ . \tag{11}$$

Therefore, the Hubble parameter during the "slow-roll" period is found from (2) to be equal to

$$H \approx \frac{m^2}{3} t \ . \tag{12}$$

A useful check at this point is to compute the scale factor R. We find

$$H \equiv \frac{\dot{R}}{R} \approx \frac{m^2}{3} t \quad \rightarrow \quad R \propto e^{\frac{m^2 t^2}{6}} = e^{\frac{Ht}{2}} \ , \tag{13}$$

as it should.

3. We are now asked to compute the Hubble parameter when the potential and kinetic terms are of the same order of magnitude. To do so, it is more convenient to work with the original system of equation for the inflaton.

   From eq. (2), we see that

$$\dot{\phi}^2 + m^2 \phi^2 = \frac{3H^2}{4\pi G} \ . \tag{14}$$

   We notice that the above implies the following change of variables

$$\dot{\phi} = \sqrt{\frac{3}{4\pi G}} H \sin\theta \ , \tag{15}$$

$$\phi = \frac{1}{m} \sqrt{\frac{3}{4\pi G}} H \cos\theta \ . \tag{16}$$

   Combining these two equations, we find

$$\dot{H} \cos\theta - H\dot{\theta} \sin\theta = mH \sin\theta \ . \tag{17}$$

   We now turn to eq. (1)

$$\ddot{\phi} + 3H\dot{\phi} + m^2 \phi = 0 \ . \tag{18}$$

Using

$$\ddot{\phi} = \frac{1}{2\dot{\phi}}\frac{d}{dt}\dot{\phi}^2 \quad \text{and} \quad \phi = \frac{1}{2\dot{\phi}}\frac{d}{dt}\phi^2 \ , \tag{19}$$

the above becomes

$$\frac{d}{dt}\dot{\phi}^2 + 6H\dot{\phi}^2 + m^2\frac{d}{dt}\phi^2 = 0 \ . \tag{20}$$

Plugging the expressions for $\dot{\phi}$ and ϕ from (15) and (16), we find after some algebra

$$\dot{H} = -3H^2 \sin^2\theta \ . \tag{21}$$

Finally, we replace the result for $\dot{H}$ into (17) to obtain

$$\dot{\theta} \approx -m \ , \tag{22}$$

where we dropped the oscillatory term, since $H \ll m$. The above expression gives us

$$\theta \approx -mt \ , \tag{23}$$

therefore

$$\dot{H} = -3H^2 \sin^2(mt) \ . \tag{24}$$

Finally,

$$\frac{1}{3H} = \frac{t}{2}\left(1 - \frac{\sin(2mt)}{2mt}\right) \approx \frac{t}{2} \ , \tag{25}$$

for $mt \gg 1$. Therefore

$$H \approx \frac{2}{3t} \ . \tag{26}$$

This expression suggests that we are inside a matter dominated period.

4. We have seen that the temperature T is related to the energy density ρ as

$$\rho = \frac{\pi^2}{30} g_*(T) T^4 \ ,$$

where $g_*(T)$ is the number of relativistic degrees of freedom at a given temperature.

We are asked to estimate the temperature at the end of inflation, i.e. when the slow roll condition is saturated :

$$\dot{\phi}^2_{end} \approx m^2 \phi^2_{end} \ , \tag{27}$$

where we denoted ϕ_{end} the value of the field at the end of inflation. This means that the energy density will be

$$\rho(\phi_{end}) = \frac{1}{2}\left(\dot{\phi}_{end}^2 + m^2\phi_{end}\right) \approx m^2\phi_{end}^2 \ . \tag{28}$$

Since we want all the energy to be transferred to the SM particles, $g_*(T_{end}) \sim 100$, therefore

$$T_{end} \approx \frac{1}{2}\sqrt{m\phi_{end}} \ . \tag{29}$$

In part 2 we saw that during slow-roll, $\dot{\phi} \approx \frac{m}{\sqrt{12\pi G}}$, so from (27) we find that

$$\phi_{end} \approx \frac{1}{\sqrt{12\pi G}} \ . \tag{30}$$