TMP-TC2: Cosmology

Problem Set 11 4 & 6 July 2023

1. Equations of motion for a scalar field in FLRW

Consider the homogeneous solutions of a scalar field ϕ described by the action

$$S[\phi] = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] ,$$

which evolves in a flat FLRW universe with the metric $ds^2 = -dt^2 + a(t)^2\mathbf{dx}^2$. Derive the equations on motion :

$$\begin{cases} \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \\ H^2 = \dfrac{8\pi G}{3}\left(\dfrac{1}{2}\dot{\phi}^2 + V(\phi)\right) \end{cases} .$$

You can use results from Sheet 1.

2. Scalar field in FLRW spacetime

Consider a generic scalar field $\phi(t, \mathbf{x})$ that evolves in a flat FLRW spacetime with the metric $ds^2 = dt^2 - a^2(t)d\mathbf{x}^2$

1. Compare the energy-momentum tensor of the scalar field with that of a perfect fluid and determine the field's density ρ and pressure p. Find $w \equiv p/\rho$.
2. Determine the condition for accelerated expansion.
3. Assume that the field is homogeneous. Starting from the continuity equation, derive its equation of motion.

3. Behaviour of the inflaton

Consider the free massive scalar field in FLRW spacetime, whose dynamics is determined by

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 ,$$
$$H^2 = \frac{8\pi G}{6}\left(\dot{\phi}^2 + m^2\phi^2\right) .$$

1. Show that the above system of equations can be reduced to a first order differential equation for $\dot{\phi}$ that reads

$$\frac{d\dot{\phi}}{d\phi} = -\frac{\dot{\phi}\sqrt{12\pi G}\sqrt{\dot{\phi}^2 + m^2\phi^2} + m^2\phi}{\dot{\phi}} .$$

2. Solve this equation and determine how the field as well as the Hubble parameter depend on time, in the following periods :
   (a) Ultra-hard, i.e. $\dot{\phi} \gg m\phi$ and $\dot{\phi}^2 \gg \frac{m^2}{\sqrt{12\pi G}}\phi$.
   (b) Slow-roll, i.e. $d\dot{\phi}/d\phi \approx 0$ and $\dot{\phi}^2 \ll m^2\phi^2$.
3. Assuming that $\dot{\phi} \sim m\phi$ and $H \ll m$, show that the Hubble parameter corresponds to the one for a matter dominated Universe

$$H = \frac{2}{3t} .$$

4. Estimate the temperature at the end of the slow-roll period, assuming that all the energy is transferred to the Standard Model particles.