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## Sheet 11:

Hand-out: Friday, Jul 07, 2023

### Problem 1 Landau parameters

In this problem we consider a fluid of fermions described by the Hamiltonian

$$\hat{\mathcal{H}} = \sum_{\mathbf{p}, \sigma} \varepsilon_{\mathbf{p}} \hat{n}_{\mathbf{p}, \sigma} + \frac{\lambda}{2} \sum_{\mathbf{p}, \sigma, \mathbf{p}', \sigma', \mathbf{q}} V(\mathbf{q}) \hat{c}_{\mathbf{p}-\mathbf{q}, \sigma}^{\dagger} \hat{c}_{\mathbf{p}'+\mathbf{q}, \sigma'}^{\dagger} \hat{c}_{\mathbf{p}', \sigma'} \hat{c}_{\mathbf{p}, \sigma}, \quad (1)$$

where  $\varepsilon_{\mathbf{p}}$  is the energy of the non-interacting Fermi gas and  $\lambda \ll 1$  is the perturbative interaction strength; Moreover,

$$V(\mathbf{q}) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} V(r) \quad (2)$$

denotes the Fourier transform of the interaction  $V(r)$ .

(1.a) Using first-order perturbation theory in  $\lambda$ , derive the Landau interaction parameters  $f_{\mathbf{p}, \mathbf{p}'}^{a, s}$  for the Fermi liquid.

(1.b) Consider the following interactions,

$$V_1(r) = \lambda_1 \delta^{(3)}(\mathbf{r}), \quad V_2(r) = -\lambda_2 \nabla^2 \delta^{(3)}(\mathbf{r}), \quad (3)$$

and calculate the Landau parameters.

(1.c) When a uniform external field (chemical potential or magnetic field) is applied, the Fermi liquid responds by becoming polarized. In addition to the free-fermion response, interactions can suppress or enhance the latter, and in extreme cases the system becomes unstable. Since the feedback of interactions is determined by the Landau parameters, the instability can be shown to occur at  $F_l^s = -1$  (Pomeranchuk instability, density response) and  $F_l^a = -1$  (Stoner instability, spin response).

Taking your results from (1.b) literally, sketch the regions of  $\lambda_{1,2}$  where the Fermi surface becomes unstable.

### Problem 2 Hubbard-Stratonovich decoupling of the Coulomb interaction - part 2

Here we consider electrons in three dimensions with mass  $m$  and Coulomb interactions as in Problem 1 on sheet 10. Our starting point is the path-integral formulation with the Hubbard-Stratonovich field  $\phi$  in Eq. (2) of sheet 10.

(3.a) Perform the fermionic Gaussian integrals  $\int \mathcal{D}[\psi^*, \psi]$  in Eq. (2) of sheet 10 and derive the effective action  $S_{\text{eff}}[\phi]$ .

- (3.b) You may now assume that the saddle-point of  $S_{\text{eff}}[\phi]$  corresponds to  $\phi \equiv 0$ . To expand  $S_{\text{eff}}$  up to quadratic order in  $\phi$ , write

$$\hat{G}^{-1} \equiv \hat{G}_0^{-1} + e\phi, \quad (4)$$

with the free electron propagator  $\hat{G}_0^{-1} = \partial_\tau + \hat{\mathbf{q}}^2/2m - \mu$ . As we are expanding around a saddle-point, the term linear in  $\phi$  vanishes. Calculate all terms of order  $\phi^2$ .

*Hint:* For operators  $\hat{A} = \hat{A}_0 + \hat{\phi}$  one may expand:

$$\text{tr} \log \hat{A} = \text{tr} \log \hat{A}_0 + \text{tr} \left( \hat{A}_0^{-1} \hat{\phi} \right) - \frac{1}{2} \text{tr} \left( \hat{A}_0^{-1} \hat{\phi} \hat{A}_0^{-1} \hat{\phi} \right) + \mathcal{O}(\hat{\phi}^3) \quad (5)$$

- (3.c) Simplify your results in (3.b) and show that the effective action takes the form

$$S_{\text{eff}}[\phi] = \sum_{\omega_n} \int \frac{d^3q}{(2\pi)^3} \left\{ -\frac{1}{2} \frac{1}{\tilde{V}(q, \omega_n)} \phi_{\mathbf{q}} \phi_{-\mathbf{q}} \right\} \quad (6)$$

with an effective screened Coulomb interaction  $\tilde{V}(q) = V(q)/\epsilon(\mathbf{q}, \omega_n)$ . Derive an expression for  $\epsilon(\mathbf{q}, \omega_n)$ .

### Problem 3 Electrons in a disordered potential

In this problem we consider free electrons in a disorder potential:

$$\hat{\mathcal{H}} = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \hat{\psi}_{\mathbf{k}}^\dagger \hat{\psi}_{\mathbf{k}} + \hat{V}_{\text{dis}}, \quad \hat{V}_{\text{dis}} = \int d^3x U(\mathbf{x}) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}). \quad (7)$$

We will perform disorder averages and assume white-noise correlations of the disorder:

$$\overline{\delta U(\mathbf{x}) \delta U(\mathbf{x}')} = g \delta(\mathbf{x} - \mathbf{x}'), \quad \text{where} \quad \delta U(\mathbf{x}) = U(\mathbf{x}) - \overline{U(\mathbf{x})} = U(\mathbf{x}) - \Delta. \quad (8)$$

- (3.a) Express the disorder potential  $\hat{V}_{\text{dis}}$  in Fourier space  $\hat{\psi}_{\mathbf{k}}^{(\dagger)}$ , and write down a diagrammatic expression for the Green's function. Explain, why the Green's function depends explicitly on two momenta.
- (3.b) Translate the diagrammatic expression for the Green's function to an algebraic one and perform the disorder average under the assumption  $\Delta = 0$ . Comment on the momentum dependence of the disorder averaged Green's function.

We interpret the second order contribution in terms of an effective interaction of the electrons and rewrite the algebraic expression of the disorder averaged Green's function as

$$\mathcal{G}(\mathbf{k}; i\omega_n) = \overline{G(\mathbf{k}, \mathbf{k}'; i\omega_n)} = \begin{array}{c} \xrightarrow{\mathbf{k}} + \xrightarrow{\mathbf{k}} \xrightarrow{\mathbf{q}} \xrightarrow{\mathbf{k}} + \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \\ \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} + \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} + \dots \end{array} \quad (9)$$

- (3.c) Reorganize this series similar to the reordering for the self energy of an interacting electronic system. Use this reordering to find a compact expression of the disorder averaged Green's function involving the "self energy"  $\Sigma(\mathbf{k}, i\omega_n)$ .

A typical approach would now include a self-consistent calculation of the self energy to ultimately calculate quantities like the spectral function. For the sake of time, we won't do this here.