Sheet 10:

Hand-out: Friday, Jun 30, 2023

Problem 1 Hubbard-Stratonovich decoupling of the Coulomb interaction - part 1

Here we consider electrons in three dimensions with mass m and Coulomb interactions

$$\hat{\mathcal{H}}_{\rm int} = \frac{1}{2}\int d^3x d^3x' \; \hat{\rho}(\boldsymbol{x})\hat{\rho}(\boldsymbol{x}')\frac{e^2}{4\pi\varepsilon_0|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{1}$$

The goal is to perform a Hubbard-Stratonovich decoupling and show that the system can be described by the path integral:

$$Z = \int \mathcal{D}[\psi^*,\psi,\phi] \; \exp\left[-\int_0^\beta d\tau \int d^3x \; \left\{\psi^*\left(\partial_\tau - \frac{1}{2m}\nabla^2 + e\phi - \mu\right)\psi - \frac{\varepsilon_0}{2}(\nabla\phi)^2\right\}\right] \tag{2}$$

(2.a) Formulate the path integral for Z starting from Eq. (1).

(2.b) Express the Coulomb interaction in Fourier modes by writing

$$\rho(\boldsymbol{x}) = \int \frac{d^3q}{(2\pi)^3} \; \rho_{\boldsymbol{q}} \; e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \tag{3}$$

and calculating $V(q)$.

(2.c) Add the auxiliary white-noise variable $\phi_{\boldsymbol{q}} = i\tilde{\phi}_{\boldsymbol{q}}$ – integrated over the imaginary axis, i.e. $\int_{-\infty}^{\infty} d\tilde{\phi}_{\boldsymbol{q}}$ in the path integral – with the contribution to the action:

$$Z_\phi = \int \mathcal{D}[\phi] \; \exp\left[-\int_0^\beta d\tau \int \frac{d^3q}{(2\pi)^3} \; \left\{-\frac{1}{2}\varepsilon_0 q^2 \phi_{\boldsymbol{q}}\phi_{-\boldsymbol{q}}\right\}\right] \tag{4}$$

Show that Z_ϕ is convergent.

(2.d) Before we apply the Hubbard-Stratonovich decoupling, consider a general *repulsive* interaction $\mathcal{H}_{\rm int} = \frac{g}{2}\sum_j A_j^2$ with $g>0$ and show that it can be replaced by $\sum_j \left(\varphi_j A_j - \frac{\varphi_j^2}{2g}\right)$ when adding the Hubbard-Stratonovich white-noise field $q_j = i\varphi_j + igA_j$.

(2.e) Continue from (2.c) and apply the technique from (2.d) to derive the path-integral in Eq. (2).

Problem 2 Feynman diagrams 1: interacting electron gas

In this problem we use the linked-cluster theorem to expand the ground state energy E_0 of an interacting electron gas to first order:

$$E_0 = iV\Sigma\{\text{linked-cluster diagrams in momentum space}\} \tag{5}$$

(2.a) Consider an interaction $V(q)$ independent of spin, and derive all first-order Feynman diagrams contributing to the linked-cluster expression for the ground state energy.

(2.b) Calculate these so-called Hartree-Fock diagrams, which consist of direct and exchange terms.

(2.c) In real-space, the first-order perturbative result for E_0 can be written

$$E_0 = \frac{1}{2}\sum_{\sigma,\sigma'}\int d^3x d^3y \; V(\boldsymbol{x}-\boldsymbol{y}) \; C_{\sigma\sigma'}(\boldsymbol{x}-\boldsymbol{y}). \tag{6}$$

Write out the corresponding real-space Feynman diagrams for E_0; next derive and calculate Feynman diagrams for the real-space correlations $C_{\sigma\sigma'}(\boldsymbol{x}-\boldsymbol{y})$.

Problem 3 Feynman diagrams 2: large-N limit

In this problem we learn a powerful approximation technique to drop certain types of diagrams. The basic idea is to consider a system with $N = 2S+1$ spin components and consider the limit $N \to \infty$ where certain classes diagrams vanish. Neglecting the same diagrams even for small values of N (down to $N = 2$) usually yields a systematic simplification of a given theory.

(2.a) Consider interacting fermions with $N = 2S+1$ spin-degeneracy and interaction strength $V(\boldsymbol{q}) = \frac{1}{N}U(\boldsymbol{q})$. Draw the Feynman diagrams expansion for the ground state energy and identify leading and sub-leading terms in the $1/N$ expansion.

(2.b) Discuss which classes of diagrams in the linked-cluster expansion of the ground state energy vanish.

(2.c) $N\chi^{(0)}(q) = \langle \delta\rho(q)\delta\rho(-q)\rangle_0$ is the susceptibility of the non-interacting Fermi gas. Draw the corresponding diagrams for the polarization bubble $\chi(q)$, up to order $1/N$ and where $q = (\boldsymbol{q}, \nu)$.

(2.d) Derive the self-energy of the system in the large-N limit and extract an effective interaction between the fermions in the large-N limit.