https://www2.physik.uni-muenchen.de/lehre/vorlesungen/sose_23/TMP-TA3/index.html

Sheet 8:

Hand-out: Friday, Jun 16, 2023

Problem 1 Residue integration

In this problem we use residue integration to calculate observables from the Green's function. We consider a free Fermi gas with spin S in continuum,

$$\hat{\mathcal{H}} = \sum_\sigma \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, \varepsilon_{\boldsymbol{k}} \hat{\psi}^\dagger_\sigma(\boldsymbol{k})\hat{\psi}_\sigma(\boldsymbol{k}), \qquad \varepsilon_{\boldsymbol{k}} = \frac{\hbar^2\boldsymbol{k}^2}{2m}, \tag{1}$$

with Fermi wavevector of length k_F.

(2.a) Show that the homogeneous density can be written as:

$$\langle\hat{\rho}(\boldsymbol{x})\rangle \equiv \sum_\sigma \langle\hat{\psi}^\dagger_\sigma(\boldsymbol{x})\hat{\psi}_\sigma(\boldsymbol{x})\rangle = -i(2S+1)\mathcal{G}(\boldsymbol{x}, t=0^-)|_{\boldsymbol{x}=0}. \tag{2}$$

and the homogeneous kinetic energy density as:

$$\langle\hat{T}(\boldsymbol{x})\rangle \equiv -\frac{\hbar^2}{2m}\sum_\sigma \langle\hat{\psi}^\dagger_\sigma(\boldsymbol{x})\boldsymbol{\nabla}^2_{\boldsymbol{x}}\hat{\psi}_\sigma(\boldsymbol{x})\rangle = i(2S+1)\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2_{\boldsymbol{x}}\mathcal{G}(\boldsymbol{x}, 0^-)|_{\boldsymbol{x}=0}. \tag{3}$$

Here $\mathcal{G}(\boldsymbol{x}, t)$ denotes the two-particle Green's function.

(2.b) Write out Fourier transforms to show that:

$$\langle\hat{\rho}(\boldsymbol{x})\rangle = (2S+1)\int \frac{d^d\boldsymbol{k}}{(2\pi)^d}\left[\int \frac{d\omega}{2\pi i} e^{i\omega\delta} \frac{1}{\omega - \varepsilon_{\boldsymbol{k}} + i\delta \mathrm{sgn}(k-k_F)}\right], \qquad \delta \to 0^+. \tag{4}$$

and find a similar expression for $\langle\hat{T}(\boldsymbol{x})\rangle$.

(2.c) Perform residue integrals to show that in $d=3$ dimensions:

$$\langle\hat{\rho}(\boldsymbol{x})\rangle = (2S+1)\frac{V_F}{(2\pi)^3}, \qquad \langle\hat{T}(\boldsymbol{x})\rangle = \frac{3}{5}\varepsilon_F\langle\hat{\rho}(\boldsymbol{x})\rangle. \tag{5}$$

Here V_F is the volume enclosed by the Fermi surface at the Fermi energy ε_F.

Problem 2 Spin coherent states

In this exercise, we introduce so-called spin-coherent states, which can be used to define path integrals of quantum-spin Hamiltonians. Spin coherent states are defined by rotating the fully polarized state $|S,S\rangle$ – with $\hat{\boldsymbol{S}}^2|S,S\rangle = S(S+1)|S,S\rangle$ and $\hat{S}^z|S,S\rangle = S|S,S\rangle$ – by angles θ around the y-axis and ϕ around the z-axis:

$$|\boldsymbol{\Omega}\rangle = e^{i\hat{S}^z\phi}\ e^{i\hat{S}^y\theta}\ e^{i\hat{S}^z\chi}|S,S\rangle. \tag{6}$$

Here $\boldsymbol{\Omega} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is a unit vector, and χ is a gauge freedom adding an overall phase. Their following properties will be useful:

$$\frac{2S+1}{4\pi}\int d\boldsymbol{\Omega}\ |\boldsymbol{\Omega}\rangle\langle\boldsymbol{\Omega}| = 1, \qquad \mathrm{tr}\hat{A} = \frac{2S+1}{4\pi}\int d\boldsymbol{\Omega}\ \langle\boldsymbol{\Omega}|\hat{A}|\boldsymbol{\Omega}\rangle \tag{7}$$

where $d\boldsymbol{\Omega} = d\theta\sin\theta d\phi$, and:

$$\langle\boldsymbol{\Omega}|\boldsymbol{\Omega}'\rangle = \left(\frac{1+\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}'}{2}\right)^S e^{-iS\psi}, \quad \psi = 2\arctan\left[\tan\left(\frac{\phi-\phi'}{2}\right)\frac{\cos[(\theta+\theta')/2]}{\cos[(\theta-\theta')/2]}\right]+\chi-\chi'. \tag{8}$$

(3.a) Construct the spin-coherent path integral, i.e. show that:

$$Z \equiv \mathrm{tr}\mathcal{T}_\tau e^{-\int_0^\beta d\tau\ \hat{\mathcal{H}}(\tau)} = \int \mathcal{D}\boldsymbol{\Omega}(\tau)\ \exp\left(-\mathcal{S}[\boldsymbol{\Omega}(\tau)]\right) \tag{9}$$

with the action:

$$\mathcal{S}[\boldsymbol{\Omega}(\tau)] = -iS\sum_j \omega[\boldsymbol{\Omega}_j] + \int_0^\beta d\tau\ \langle\boldsymbol{\Omega}(\tau)|\hat{\mathcal{H}}(\tau)|\boldsymbol{\Omega}(\tau)\rangle. \tag{10}$$

Here j labels different spins in a lattice, and we defined

$$\exp\left[iS\omega[\boldsymbol{\Omega}]\right] = \prod_{n=1}^{N}\langle\boldsymbol{\Omega}(\tau_n+\delta\tau)|\boldsymbol{\Omega}(\tau_n)\rangle, \qquad \tau_n = n\delta\tau,\ \delta\tau = \beta/N. \tag{11}$$

Which boundary conditions does $\boldsymbol{\Omega}(\tau)$ in the path integral obey?

(3.b) Simplify the contribution $\omega[\boldsymbol{\Omega}]$ to the action by assuming continuous differentiable trajectories and show that:

$$\omega[\boldsymbol{\Omega}] = -\int_0^\beta d\tau\ (\partial_\tau\phi)\cos[\theta] + \partial_\tau\chi. \tag{12}$$

By choosing the gauge convention $\chi_j(\tau) \equiv 0$, simplify the result further and show that:

$$\omega[\boldsymbol{\Omega}] = -\oint_{\phi_0}^{\phi_\beta} d\phi\ \cos(\theta). \tag{13}$$

Discuss how this Berry-phase contribution is geometric and does not depend on the explicit time-dependence of $\phi(\tau)$.

(3.c) Construct the effective action S for a Heisenberg interaction:

$$\hat{\mathcal{H}} = J \sum_{\langle i,j \rangle} \hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_j. \tag{14}$$

Hint: Show first that $\boldsymbol{\Omega} \cdot \hat{\boldsymbol{S}} |\boldsymbol{\Omega}\rangle = S |\boldsymbol{\Omega}\rangle$.

(3.d) Discuss on generic grounds and using the above path-integral formalism why $S \to \infty$ corresponds to the classical limit.