https://www2.physik.uni-muenchen.de/lehre/vorlesungen/sose_23/TMP-TA3/index.html

Sheet 1:

Hand-out: Friday, Apr. 21, 2023

Problem 1 Berry phases

In an adiabatic evolution along a closed loop $\mathcal{C} : \boldsymbol{g}(t)$ in time $t = 0...T$, an eigenstate $|\Psi_n(\boldsymbol{g})\rangle$ with energy $E_n(\boldsymbol{g})$ picks up a geometric (Berry-) and a dynamical phase:

$$|\Phi(T)\rangle = e^{i\left(\varphi_{\rm B} - \varphi_{\rm dyn}(T)\right)} |\Phi(0)\rangle, \qquad |\Phi(0)\rangle = |\Psi_n(\boldsymbol{g}(0))\rangle, \tag{1}$$

where:

$$\varphi_{\rm B} = \oint_{\mathcal{C}} d\boldsymbol{g} \cdot \langle \Psi_n(\boldsymbol{g}) | i \boldsymbol{\nabla}_{\boldsymbol{g}} | \Psi_n(\boldsymbol{g}) \rangle, \qquad \varphi_{\rm dyn}(T) = \int_0^T dt \; E_n(\boldsymbol{g}(t)). \tag{2}$$

(1.a) Derive the result in Eq. (2) by making an ansatz $|\Phi(t)\rangle = e^{i\varphi(t)} |\Psi_n(\boldsymbol{g}(t))\rangle$.

(1.b) The eigenstates $|\Psi_n(\boldsymbol{g})\rangle$ are only defined up to an arbitrary overall phase. Derive how the Berry connection

$$\boldsymbol{\mathcal{A}}_n(\boldsymbol{g}) = \langle \Psi_n(\boldsymbol{g}) | i \boldsymbol{\nabla}_{\boldsymbol{g}} | \Psi_n(\boldsymbol{g}) \rangle \tag{3}$$

transforms under gauge transformations

$$|\Psi_n(\boldsymbol{g})\rangle \to e^{i\vartheta_n(\boldsymbol{g})} |\Psi_n(\boldsymbol{g})\rangle, \qquad \vartheta_n(\boldsymbol{g}) \in \mathbb{R}, \tag{4}$$

and show that the Berry phase is invariant under such gauge transformations, $\varphi_B \to \varphi_B \mod 2\pi$, up to multiples of 2π.

(1.c) Consider a discrete parametrization $\boldsymbol{g}_j = \boldsymbol{g}(t = j\; T/N)$ with $j = 1...N$ which converges to $\boldsymbol{g}(t)$ when $N \to \infty$. Show that

$$\lim_{N\to\infty} \prod_{j=1}^{N} \langle \Psi_n(\boldsymbol{g}_{j+1}) | \Psi_n(\boldsymbol{g}_j) \rangle = \exp[i\varphi_{\rm B}] \tag{5}$$

where $\boldsymbol{g}_{N+1} := \boldsymbol{g}_1$. Further, show for a given $N \in \mathbb{Z}_{>0}$ that the product on the left in Eq. (5) is fully gauge invariant under $|\Psi_n(\boldsymbol{g}_j)\rangle \to e^{i\vartheta_j} |\Psi_n(\boldsymbol{g}_j)\rangle$.

(1.d) Consider a second parameter $\lambda \in [0, 1]$, such that $\mathcal{M} : \boldsymbol{g}_\lambda$ is a parameterization of a manifold $\mathcal{M}$ in parameter space. Assuming that $\mathcal{M}$ is a simply connected two-dimensional surface, with

$$\varphi_{\rm B}(\lambda = 0) \equiv \varphi_{\rm B}(\lambda = 1) \mod 2\pi, \tag{6}$$

show that the winding number of the Berry phase defines an integer-quantized (topological) invariant (the Chern number):

$$C_{\mathcal{M}} = \frac{1}{2\pi} \int_0^1 d\lambda \; \partial_\lambda \varphi_{\rm B}(\lambda) \quad \in \quad \mathbb{Z}. \tag{7}$$

Discuss the meaning of non-zero invariants $C_{\mathcal{M}} \neq 0$.

Problem 2 Many-Body wavefunctions

In this exercise you will familiarize yourself with multi-variable many-body wavefunctions. These can contain loads of interesting physics, and the main goal here is to learn some of their general properties and work with some explicit examples.

(2.a) Show (as a warm-up, independent of the next problems) that the normalization of occupation number states introduced in the lecture,

$$|\{n_{\boldsymbol{r}}\}_{\boldsymbol{r}}\rangle = \mathcal{N}_{\pm}\hat{S}_{\pm}|\boldsymbol{r}_1, ..., \boldsymbol{r}_N\rangle, \tag{8}$$

is given by:

$$\mathcal{N}_{\pm} = \left[\frac{N!}{\prod_{\boldsymbol{r}} n_{\boldsymbol{r}}!}\right]^{-1/2} \tag{9}$$

(2.b) Consider a one-dimensional system of N bosons, with coordinates $x_1, ..., x_N$. Explain why it is sufficient to know the many-body wavefunction $\Psi_1(x_1, ..., x_N)$ on the subset $x_1 < x_2 < ... < x_N$. From Ψ_1 construct the full bosonic wavefunction $\Psi_+(x_1, ..., x_N)$ for arbitrary $x_1, ..., x_N$ by summing over all permutations P of $j = 1, ..., N$.

(2.c) *Bose-Fermi mapping in 1D*: Following the procedure in (2.b), construct a full fermionic wavefunction $\Psi_-(x_1, ..., x_N)$ for arbitrary $x_1, ..., x_N$ by summing over all permutations P of $j = 1, ..., N$. Find a general relation between $\Psi_+(x_1, ..., x_N)$ and $\Psi_-(x_1, ..., x_N)$.

(2.d) *Lieb-Liniger gas*: Consider the one-dimensional Lieb-Liniger gas of bosons described by the Hamiltonian

$$\hat{\mathcal{H}}_{\rm LL} = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{1}{2}\sum_{i\neq j}\Phi_0\delta(x_i - x_j). \tag{10}$$

Show that (i) within the domain $x_1 < x_2 < ... < x_N$ introduced in (2.b), the eigenstates are plane waves, i.e. up to normalization $\Psi_1(x_1, ..., x_N) = \prod_{j=1}^{N} e^{ik_j x_j}$; then show (ii) that at the boundary of the domain, the following condition must be satisfied:

$$\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\Psi_+|_{x_i - x_j = 0^+} = -\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\Psi_+|_{x_i - x_j = 0^-} = \frac{m\Phi_0}{\hbar^2}\Psi_+|_{x_i - x_j = 0^\pm}, \tag{11}$$

where the discontinuity in the derivative is proportional to the interaction strength Φ_0; Finally, (iii) derive a similar condition for the fermionic counterpart $\Psi_-(x_1, ..., x_N)$: show that it has a discontinuity in the wavefunction which is inverse proportional to Φ_0.

Your results show that weakly interacting bosons map to strongly interacting fermions in 1D, and vice-versa.