Problem 1

Take the bidoublet

$$\Phi = (\tilde{\varphi} \;\; \varphi) \qquad , \qquad (1)$$

with

$$\varphi = \begin{pmatrix} \varphi^+ \\ \varphi \end{pmatrix} \qquad , \qquad (2)$$

and

$$\tilde{\varphi} = i\sigma_2 \varphi^* = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \varphi^- \\ \varphi^0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \varphi^- \\ \varphi^0 \end{pmatrix} \qquad (3)$$

$$= \begin{pmatrix} (\varphi^0)^* \\ -\varphi^- \end{pmatrix}$$

Thus, explicit form of Φ is

$$\Phi = \begin{pmatrix} (\varphi^0)^* & \varphi^+ \\ -\varphi^- & \varphi^0 \end{pmatrix} \qquad (4)$$

(i) Take (4) & find its hermitian conjugate Φ^{+}

$$\Phi^{+} = \begin{pmatrix} \varphi^{0} & -\varphi^{+} \\ \varphi^{-} & (\varphi^{0})^{*} \end{pmatrix} \qquad (5)$$

From (4), (5) we get

$$Tr(\Phi^{+}\Phi) = Tr\left[\begin{pmatrix} \varphi^{0} & -\varphi^{+} \\ \varphi^{-} & (\varphi^{0})^{*} \end{pmatrix}\begin{pmatrix} (\varphi_{0})^{*} & \varphi^{+} \\ -\varphi^{-} & \varphi^{0} \end{pmatrix}\right] \qquad (6)$$

$$= 2\left(|\varphi^{0}|^{2} + |\varphi^{+}|^{2}\right)$$

At the same time

$$\varphi^{+}\varphi = |\varphi^{0}|^{2} + |\varphi^{+}|^{2} \qquad (7)$$

,

therefore

$$Tr(\Phi^{+}\Phi) = \frac{1}{2}\varphi^{+}\varphi \quad . \qquad (8)$$

(ii) We can write (8) in term of components say f_i, $i = 1, \ldots, 4$.

Then

$$\mathrm{Tr}(\Phi^\dagger \Phi) = \sum_i f_i^2 \rightarrow SO(4) \text{ symmetry.} \quad (9)$$

since $SO(4) \simeq SU(2)_1 \times SU(2)_2$, we have

$$\Phi \rightarrow \Phi' = U_1 \Phi U_2^\dagger \quad (10)$$

Let's check that, indeed, the trace is invariant wrt (10):

$$\mathrm{Tr}(\Phi^\dagger \Phi) \rightarrow \mathrm{Tr}(\Phi^{\dagger\prime} \Phi') = \mathrm{Tr}[U_2 \Phi^\dagger \overbrace{U_1^\dagger U_1}^{\mathbb{1}} \Phi U_2^\dagger]$$

$$= \mathrm{Tr}(\Phi^\dagger \Phi) \quad ✓ \quad (11)$$

$= \mathbb{1}$

cyclic property.

(iii) $$\det \Phi \rightarrow \det \Phi' = \det U_1 \Phi U_2^\dagger$$

$$= \det U_1 \det \Phi \det U_2^\dagger$$

$$= \det \Phi \quad ✓ \quad (12)$$

(iv) we have at our disposal the following invariants:

$$\mathrm{Tr}(\Phi^\dagger\Phi), \quad \det\Phi, \quad \det\Phi^\dagger, \quad \mathrm{Tr}(\Phi^\dagger\Phi\Phi^\dagger\Phi)$$

$\det\Phi^\dagger \nearrow$ h.c. of $\det\Phi$

⊛ <u>HOWEVER</u> $\det\Phi$ & $\mathrm{Tr}(\Phi^\dagger\Phi)$ are not independent:

$$\det\Phi = \det\begin{pmatrix} (\varphi^0)^* & \varphi^+ \\ -\varphi^- & \varphi^0 \end{pmatrix} = |\varphi_0|^2 + |\varphi^+|^2$$

$$= \frac{1}{2}\mathrm{Tr}(\Phi^\dagger\Phi) \qquad (13)$$

conclusion : we don't need to include $\det\Phi$, $\det\Phi^\dagger$ in the potential.

⊛ what about $\mathrm{tr}(\Phi^\dagger\Phi\Phi^\dagger\Phi)$? Certainly it is invariant. BUT, we can explicitely see that

$$\mathrm{Tr}(\Phi^\dagger\Phi\Phi^\dagger\Phi) = 2(|\varphi^0|^2 + |\varphi^+|^2)^2$$

$$= \frac{1}{2}\mathrm{Tr}(\Phi^\dagger\Phi)^2 \qquad (14)$$

Thus, the most general potential reads

$$V(\Phi) = m^2 \, Tr(\Phi^\dagger \Phi) + \lambda \, Tr(\Phi^\dagger \Phi)^2 . \qquad (15)$$

Problem 2

Take now

$$\Phi = (\tilde{\varphi}_1 \quad \varphi_2) \quad , \qquad (1)$$

with

$$\varphi_i = \begin{pmatrix} \varphi_i^+ \\ \varphi_i^0 \end{pmatrix} \quad , i = 1, 2 \quad , \qquad (2)$$

& as before

$$\tilde{\varphi}_i = i \sigma_2 \varphi_i^* = \begin{pmatrix} (\varphi_i^0)^* \\ -\varphi_i^- \end{pmatrix} . \qquad (3)$$

therefore

$$\Phi = \begin{pmatrix} (\varphi_1^0)^* & \varphi_2^+ \\ -\varphi_1^- & \varphi_2^0 \end{pmatrix} \quad , \qquad (4)$$

and

$$\Phi^{+} = \begin{pmatrix} \varphi_1^0 & -\varphi_1^+ \\ \varphi_2^- & (\varphi_2^0)^* \end{pmatrix} . \quad (5)$$

(i) From (4) & (5), we get

$$Tr(\Phi^+\Phi) = |\varphi_1^0|^2 + |\varphi_1^+|^2 + |\varphi_2^0|^2 + |\varphi_2^+|^2 , \quad (6)$$

which is manifestly SM-invariant.

(ii) as before, let's write in term of components the above f_i, $i = 1, \dots, 8$.

It becomes clear that

$$Tr(\Phi^+\Phi) = \sum_{i=1}^{8} f_i^2 , \quad (7)$$

meaning that it is $SO(8)$-invariant.

(iii), (iv) we have the following possibilities:

$Tr(\Phi^+\Phi)$, $Tr(\Phi^+\Phi\Phi^+\Phi)$, $\det\Phi$, $\det\Phi^+$ (8)

let's check what we get:

- $Tr(\Phi^+\Phi\Phi^+\Phi) \neq Tr(\Phi^+\Phi)^2$, (8)

actually it also depends on $\det\Phi$.

Thus,

$$V(\Phi) = m_1^2 \, \mathrm{Tr}(\Phi^\dagger \Phi) + m_2^2 (\det \Phi + h.c.) \qquad (9)$$
$$+ \lambda_1 \, \mathrm{Tr}^2(\Phi^\dagger \Phi) + \lambda_2 \, \mathrm{Tr}\, \Phi^\dagger \Phi \, (\det \Phi + h.c.)$$

(v) Define

$$\tilde{\Phi} = i\sigma_2 \Phi^* i\sigma_2 \, . \qquad (10)$$

Then

$$\tilde{\Phi} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1^0 & \varphi_2^- \\ -\varphi_1^+ & (\varphi_2^0)^* \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\varphi_1^+ & (\varphi_2^0)^* \\ -\varphi_1^0 & -\varphi_2^- \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (11)$$

$$= \begin{pmatrix} -(\varphi_2^0)^* & -\varphi_1^+ \\ \varphi_2^- & -\varphi_1^0 \end{pmatrix} \overset{(3)}{=} -(\tilde{\varphi}_2 \;\; \varphi_1)$$

(vi) $$\Phi^\dagger \Phi + \tilde{\Phi}^\dagger \tilde{\Phi} = \begin{pmatrix} |\varphi_1^0|^2 + |\varphi_1^+|^2 + |\varphi_2^0|^2 + |\varphi_2^+|^2 & 0 \\ 0 & \end{pmatrix} \qquad (12)$$

manifestly invariant under $SU(2) \times SU(2)$.

(vii) $\mathrm{Tr}(\Phi^+\tilde{\Phi}) = -2\left(\varphi_1^0(\varphi_2^0)^* + \varphi_1^+(\varphi_2^+)^*\right)$ (13)

comparing with

$$\det\Phi^+ = \varphi_1^0(\varphi_2^0)^* + \varphi_1^+(\varphi_2^+)^* , \quad (14)$$

we obtain

$$\mathrm{Tr}(\Phi^+\tilde{\Phi}) = -2\det\Phi^+ . \quad (15)$$

(viii) We have

$$\mathrm{Tr}(\Phi^+\Phi),\ \mathrm{Tr}\,\tilde{\Phi}^+\Phi + h.c.,\ \ldots$$

Problem 3

Take

$$V = -\frac{\mu^2}{2}(\varphi_L^2+\varphi_R^2) + \frac{\lambda}{4}(\varphi_L^4+\varphi_R^4) + \frac{\lambda'}{2}\varphi_L^2\varphi_R^2 , \quad (1)$$

$\mu, \lambda, \lambda' > 0$.

The extrema of V are determined by

$$\left.\frac{\partial V}{\partial \varphi_L}\right|_{\substack{\varphi_L=\langle\varphi_L\rangle \\ \varphi_R=\langle\varphi_R\rangle}} = \left.\frac{\partial V}{\partial \varphi_R}\right|_{\substack{\varphi_L=\langle\varphi_L\rangle \\ \varphi_R=\langle\varphi_R\rangle}} = 0 . \quad (2)$$

Explicitly, from (2) we get:

$$\langle\varphi_L\rangle\left(-\mu^2+\lambda\langle\varphi_L\rangle^2+\lambda'\langle\varphi_R\rangle^2\right)=0\ , \quad (3)$$

$$\langle\varphi_R\rangle\left(-\mu^2+\lambda\langle\varphi_R\rangle^2+\lambda'\langle\varphi_L\rangle^2\right)=0\ . \quad (4)$$

from (3) & (4), we immediately see the saddles of (1):

(i) $\langle\varphi_L\rangle=0\ ,\quad \langle\varphi_R\rangle=0\ ,$ (5)

(ii) $\langle\varphi_L\rangle=0\ ,\quad \langle\varphi_R\rangle^2=\dfrac{\mu^2}{\lambda}\neq 0\ ,$ (6)

(iii) $\langle\varphi_L\rangle^2=\dfrac{\mu^2}{\lambda}\neq 0\ ,\quad \langle\varphi_R\rangle=0\ ,$ (7)

(iv) $\langle\varphi_L\rangle^2=\langle\varphi_R\rangle^2=\dfrac{\mu^2}{\lambda+\lambda'}\neq 0\ .$ (8)

whether (i)-(iv) correspond to minima or maxima is found by inspecting the mass matrix (2^{nd} derivatives of V)

$$\frac{\partial^2 V}{\partial\varphi_i\varphi_j}=\begin{pmatrix}-\mu^2+3\lambda\langle\varphi_L\rangle^2+\lambda'\langle\varphi_R\rangle^2 & 2\lambda'\langle\varphi_L\rangle\langle\varphi_R\rangle \\ 2\lambda'\langle\varphi_L\rangle\langle\varphi_R\rangle & -\mu^2+3\lambda\langle\varphi_R\rangle^2+\lambda'\langle\varphi_L\rangle^2\end{pmatrix} \quad (9)$$

evaluated on (i)-(iv).

(i) Take (5) & plug into (9):

$$\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}(i) = -\mu^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} < 0 , \quad (10)$$

meaning that this is maximum.

(ii) Take (6) & plug into (9):

$$\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}(ii) = \mu^2 \begin{pmatrix} \frac{\lambda' - \lambda}{2} & 0 \\ 0 & 2 \end{pmatrix} , \quad (11)$$

meaning that (ii) is a minimum for

$$\lambda' > \lambda \qquad . \quad (12)$$

(iii) same as (ii), obviously.

(iv) Take (8) & plug into (9):

$$\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}(iv) = \frac{2\mu^2}{\lambda + \lambda'} \begin{pmatrix} \lambda & \lambda' \\ \lambda' & \lambda \end{pmatrix} . \quad (13)$$

The eigenvalus are

$$m_1^2 = 2\mu^2, \quad m_2^2 = 2\mu^2 \frac{\lambda - \lambda'}{\lambda + \lambda'} , \quad (14)$$

so requiring $m_1^2 > 0$, $m_2^2 > 0$, we get

$$\lambda > \lambda' \quad , (15)$$

for (ii) to be a minimum.

What happens for $\lambda = \lambda'$?

$$V = -\frac{\mu^2}{2}(\varphi_L^2 + \varphi_R^2) + \frac{\lambda}{4}(\varphi_L^4 + \varphi_R^4 + 2\varphi_L\varphi_R)$$

$$= -\frac{\mu^2}{2}(\varphi_L^2 + \varphi_R^2) + \frac{\lambda}{4}(\varphi_L^2 + \varphi_R^2)^2 \quad . (16)$$

$$= -\mu^2 X^+ X + \lambda (X^+ X)^2 \quad , \rightarrow U(1) \text{ symmetry.}$$

with

$$X = \varphi_L + i\varphi_R \quad . \qquad (17)$$

Problem 4

① Up to dimension four, there are the following invariants :

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(a) $\mathrm{Tr}(\Delta^\dagger \Delta)$, (1)

(b) $\mathrm{Tr}^2(\Delta^\dagger \Delta)$, (2)

(c) $\mathrm{Tr}(\Delta^2)\mathrm{Tr}(\Delta^\dagger)^2$, (3)

(d) $\mathrm{Tr}(\Delta^\dagger \Delta \Delta^\dagger \Delta)$. (4)

[why do I not include $\det \Delta$, $\det \Delta^\dagger$?]

So, (d) is not independent:

$$\mathrm{Tr}(\Delta^\dagger \Delta \Delta^\dagger \Delta) = \mathrm{Tr}^2(\Delta^\dagger \Delta) - \frac{1}{2}\mathrm{Tr}(\Delta^2)\mathrm{Tr}(\Delta^\dagger)^2 \quad (5)$$

Therefore

$$V = -\mu^2 \mathrm{Tr}(\Delta^\dagger \Delta) + \lambda_1 \mathrm{Tr}^2(\Delta^\dagger \Delta) + \lambda_2 \mathrm{Tr}\Delta^2 \mathrm{Tr}\Delta^{\dagger 2} . \quad (6)$$

② Take

$$\Delta = \Delta_1 + i\Delta_2 , \quad (7)$$

with

Δ_1, Δ_2 = Hermitian, i.e. $\Delta_i^\dagger = \Delta_i$ (8)

Now $\Delta \to U \Delta U^\dagger = U\Delta_1 U^\dagger + i U \Delta_2 U^\dagger$. (9)

Choose U such that Δ_1 becomes diagonal & real, i.e.

$$\Delta_1 = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} = a\sigma_3 \quad , a \in \mathbb{R} \quad , \quad (10)$$

and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. (11)

Thus,

$$\Delta = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} + i U \Delta_2 U^{+} . \quad (12)$$

Since U diagonalized Δ_1, it cannot diagonalize Δ_2:

$$\Delta = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} + i \begin{pmatrix} b & re^{-i\phi} \\ re^{i\phi} & -b \end{pmatrix} , \quad (13)$$

with $b, r, \phi \in \mathbb{R}$. Since, however, $\Delta_1 \propto \sigma_3$, we can rotate both terms by

$$U_3 = \begin{pmatrix} e^{i\vartheta} & 0 \\ 0 & e^{-i\vartheta} \end{pmatrix} , \quad (14)$$

which will only affect the 2nd term:

$$\Delta = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} + i \begin{pmatrix} b & re^{-i(\varphi-2\theta)} \\ re^{i(\varphi-2\theta)} & -b \end{pmatrix} \quad (15)$$

meaning that for $\theta = \varphi/2$, we have

$$\Delta = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} + i \begin{pmatrix} b & r \\ r & -b \end{pmatrix} . \quad (16)$$

③ Let's take Δ from (16)

$$\Delta = \begin{pmatrix} z & ir \\ ir & -z \end{pmatrix} , \quad (17)$$

with

$$z = a + ib . \quad (18)$$

Plug (17) into the potential:

$$V = -2\mu^2 (|z|^2 + r^2) + 4\lambda_1 (|z|^2 + r^2)^2 + 4\lambda_2 (z^2 - r^2)(z^{*2} - r^2) . \quad (18)$$

For $\lambda_2 > 0$, we find

$$z = r \qquad , \qquad (19)$$

meaning that

$$\Delta = r \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \qquad . \quad (20)$$

4 The matrix is very easy to find by taking

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \qquad , \quad (21)$$

and plugging into

$$U \langle \Delta \rangle U^\dagger \qquad . \quad (22)$$

5 $Q = T_3 + \frac{B-L}{2}$. (21)

Then,

$$Q \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} = \frac{1}{2}\left[\sigma_3 , \frac{v}{2}(\sigma_1 - i\sigma_2)\right] + \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \quad (22)$$

$$= v\left(i\sigma_2 - \sigma_1 + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = 0 \ .$$

6 Starting from

$$D_\mu \langle\Delta\rangle = -ig\,\bar{B}_\mu \langle\Delta\rangle - ig\,A_\mu^R [T_R, \langle\Delta\rangle], \quad (23)$$

we easily find

$$\mathrm{Tr}\left(D_\mu \langle\Delta\rangle^\dagger D_\mu \langle\Delta\rangle\right) = g^2 v^2 W_R^+ W_R^- + (g^2+\bar{g}^2) v^2 Z_R^2, \quad (24)$$

with

$$Z_\mu^R = \frac{1}{\sqrt{g^2+\bar{g}^2}} \left(g A_\mu^{3R} - \bar{g}\,\bar{B}_\mu\right) . \quad (25)$$

Introducing

$$\sin\theta_R = \frac{\bar{g}}{\sqrt{g^2+\bar{g}^2}}, \quad \cos\theta_R = \frac{g}{\sqrt{g^2+\bar{g}^2}}, \quad (26)$$

eq. (25) yields

$$Z_\mu^R = \cos\theta_R\, A_\mu^{3R} - \sin\theta_R\, \bar{B}_\mu . \quad (27)$$

7 $$\frac{m_{Z_R}^2}{m_{W_R}^2} = 2\,\frac{g^2+\bar{g}^2}{g^2} = 2\left(1+\frac{\bar{g}^2}{g^2}\right) . \quad (28)$$

Since

$$\frac{1}{g'^2} = \frac{1}{g^2} + \frac{1}{\bar{g}^2} \quad , \quad (29)$$

we get

$$\frac{g^2}{\bar{g}^2} = \frac{g^2}{g'^2} - 1 = \frac{1}{\tan^2\theta_W} - 1 \quad , \quad (30)$$

meaning that

$$\frac{m_{Z_R}^2}{m_{W_R}^2} = \frac{2}{1-\tan^2\theta_W} \quad . \quad (31)$$

8 Take $\lambda_2 < 0$, meaning that

$$\langle \Delta \rangle = \begin{pmatrix} 0 & ir \\ ir & 0 \end{pmatrix} = ir\sigma_1 \quad , \quad (32)$$

thus

$$[\sigma_1, \langle \Delta \rangle] = 0 \quad . \quad (33)$$