LMU "Introduction to Physics of Neutrinos" Course 2023

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Homework 3

Problem 1

Take a non-Abelian gauge theory based on the group SU(N). Construct all possible invariants using:

- A scalar field Φ in the fundamental representation. Recall the transformation property $\Phi \to U\Phi$ under SU(N).
- A scalar field Σ in the adjoint representation. Recall $\Sigma \to U\Sigma U^\dagger$ under SU(N), and $\Sigma = \Sigma^\dagger$, $Tr\Sigma = 0$.
- Both Φ and Σ.
- As you know, we truncate potentials at fourth order in the field(s), for the sake of renormalizability.

  Take SU(2) and SU(3) examples to discuss explicitly the three cases above for the quartic potentials.

  Now assume a discrete symmetry $\Sigma \to -\Sigma$ in the SU(3) case. What happens - what is the true symmetry of the potential?

Problem 2

Study case of the $SO(3)$ symmetry breaking with scalar triplet representations.

Part I

In this case we have a quartic potential

$$V = -\frac{\mu^2}{2}\Phi^T\Phi + \frac{\lambda}{4}\left(\Phi^T\Phi\right)^2 \ , \tag{1}$$

where Φ is a 3-vector (column) transforming as

$$\Phi \to O\Phi, \quad O^TO = OO^T = 1 \ . \tag{2}$$

Using the SO(3) symmetry, we can always rotate the ground state value (vev) to be in the third direction, so that we can write

$$\Phi = \begin{pmatrix} G_1 \\ G_2 \\ v+h \end{pmatrix}, \tag{3}$$

where the vev v is the value of field at the minimum of the potential.

- Show that $\mu^2 = \lambda v^2$. What is the vacuum manifold, i.e. the manifold of all values of Φ_0 that minimize the potential?

- Show that $G_{1,2}$ are massless Nambu-Goldstone (NG) bosons, and that h is the analog of the Higgs boson for the local gauge case, with $m_h^2 = 2\lambda v^2$.

- Show that the generators $T_{1,2}$ are broken, unlike T_3 (use $(T_i)_{jk} = -i\epsilon_{ijk}$ for the vector representation).

  This establishes the correspondence between the number of massless Nambu-Goldstone bosons and the number of broken genearators.

- Let us now gauge the theory by promoting the global SO(3) to a local symmetry. Write down the corresponding Lagrangian.
  Hint: You should generalize the partial derivative to a covariant one and add a kinetic term for the gauge fields.

- Find the mass spectrum of the gauge fields by expanding the appropriate terms of the Lagrangian around the vacuum that you found before.

Part II

Take *two* such scalar triplets Φ_1 and Φ_2 and assume the additional discrete symmetries

$$\begin{aligned} D_1 &: \Phi_1 \longrightarrow -\Phi_1 \qquad \Phi_2 \longrightarrow \Phi_2 \ , \\ D_2 &: \Phi_1 \longrightarrow \Phi_1 \qquad \Phi_2 \longrightarrow -\Phi_2 \ , \end{aligned}$$

which implies the following most general potential

$$V = -\sum_{i=1}^{2} \frac{\mu_i^2}{2} \Phi_i^T \Phi_i + \sum_{i=1}^{2} \frac{\lambda_i}{4} \left(\Phi_i^T \Phi_i\right)^2 + \frac{\lambda_3}{2} \Phi_1^T \Phi_1 \Phi_2^T \Phi_2 + \frac{\lambda_4}{2} \left(\Phi_1^T \Phi_2\right)^2 \ . \tag{4}$$

1. Find the possible patterns of symmetry breaking, *i.e.* determine the resulting symmetry groups.

2. In each case, compute the particle spectrum and determine the massless NG bosons. In the case of non-diagonal 2×2 mass matrices, it is sufficient that you write the conditions for the positivity of $(masses)^2$, i.e. you do not have to find the eigenstates and eigenvalues.

3. Assume the symmetry to be gauged and find the gauge boson masses for respective patterns of symmetry breaking.

Advice: give it a thought before plunging into calculating. Use symmetry-like arguments - it will help tremendously if you find the right one :). If the analysis of symmetry breaking gets too messy, then you are somehow doing something wrong or not seeing what is going on.

Problem 3

Consider a theory of three complex scalar fields $\varphi_i,\ i = 1, 2, 3$, with Lagrangian

$$\mathcal{L} = \partial_\mu \varphi_i^* \partial^\mu \varphi_i + \mu^2 \varphi_i^* \varphi_i - \lambda(\varphi_i^* \varphi_i)^2 \ .$$

1. Find the symmetry group of the above.
2. Minimize the potential and determine the ground states of the system. Choosing one of them, find the unbroken subgroup.
3. By considering perturbations on top of the ground state find the Nambu-Goldstone mode(s) and the masses of the rest of the perturbations.