LMU "Introduction to Physics of Neutrinos" Course 2023
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Homework 2

Problem 1

Consider the Proca Lagrangian which describes a massive $U(1)$ gauge field, consisting of the Maxwell Lagrangian and the mass term

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A^2 A_\mu A^\mu \ . \tag{1}$$

1. Derive the equation of motion for the Proca field A_μ.

2. Prove that on-shell (equation of motion) $\partial_\mu A^\mu = 0$. Discuss the number of degrees of freedom and explain them. What are the physical states, i.e. what are they characterized with?

3. Compute the propagator for the massive gauge field. Discuss the high energy limit.

   Hint: Define the propagator as the inverse of the quadratic term in momentum space.

4. Take instead the Maxwell theory, defined as $m_A = 0$ in (1). Show that one *cannot* construct a propagator. Add a gauge fixing term

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2\xi} (\partial_\mu A^\mu)^2 \ , \tag{2}$$

   and find the ξ-dependent propagator.

5. Couple the Proca field to a chiral current, i.e. consider the following interaction term

$$\mathcal{L}_{A,int} = g \, \bar{\psi}_L \gamma_\mu \psi_L \, A^\mu, \tag{3}$$

   where ψ denotes a massive fermion with mass m_f and usual kinetic terms for ψ_L and ψ_R. In other words, only ψ_L carries the $U(1)$ charge in question.

   Compute the A exchange for the $\psi - \psi$ scattering as a function of the exchange momentum q and particle masses. Do not worry about signs and the factors of i - just get the form of the interaction through the A propagator computed above.

   Discuss the limit $m_A \to 0$, with finite m_f. Is it singular?

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Problem 2

Instead of Proca, consider now the so-called Stückelberg Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_A^2(A_\mu - \frac{1}{m_A}\partial_\mu G)(A^\mu - \frac{1}{m_A}\partial^\mu G) \ . \qquad (4)$$

1. Prove that gauge invariance $A_\mu \to A_\mu + \partial_\mu \alpha(x)$ is restored, with a proper transformation of G.

   Find the propagators for A and G.

   Hint: Add the R_ξ gauge fixing term

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2\xi}(\partial_\mu A^\mu + \xi m_A G)^2. \qquad (5)$$

2. Show that the propagators for A and G have a good high-energy behaviour. For what value of ξ does one get the Proca propagator? Keep in mind that the gauge boson and scalar propagators have opposite overall signs.

3. Can you explain why G cannot be a physical state?

4. Add to (4) the following interaction of the field G

$$\mathcal{L}_{G,int} = m_f \ \bar{\psi}_L \psi_R \ \exp{(-ig \, G/m_A)} + h.c. \ , \qquad (6)$$

   and show that the total Lagrangian is invariant under the following gauge transformation

$$\psi_L \to \exp{(ig\alpha(x))}\psi_L \ , \quad \psi_R \to \psi_R \ , \quad A_\mu \to A_\mu + \partial_\mu \alpha(x) \ , \quad G \to G + m_A \alpha(x) \ . \quad (7)$$

   Write down the above interaction keeping only the leading term in G.

5. Compute again, as in Problem 1, the $\psi - \psi$ scattering in the R_ξ gauge, including both A and G exchanges.

   Does the result depend on ξ? Compare with the Proca calculation of the same scattering.

6. Take the limit g=0 so that the gauge boson is completely decoupled, and write down the result for the $\psi - \psi$ scattering. Explain the result on symmetry (breaking) grounds.

7. Take now the Higgs version of the U(1) gauge symmetry, with a complex charged scalar field ϕ and a quartic potential as discussed in the lecture. In other words, assume SSB, so that we can write $\phi = v + h + iG$.

   Include the Yukawa interaction

$$\mathcal{L}_Y = y_f \bar{\psi}_L \psi_R \, \phi + h.c. \ , \qquad (8)$$

   Write down the charges of the LH and RH fermion fields, and the scalar ϕ.

   Derive the interactions of the Higgs field h and G with the fermions as functions of the fermion and gauge boson masses, and compare with the Stückelberg approach.

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