LMU “Introduction to Physics of Neutrinos” Course 2023
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Homework 1

Problem 1

A four-component Dirac spinor transforms under the Lorentz group as

$$\Psi \to S\,\Psi, \qquad S \equiv e^{i\theta_{\mu\nu}\Sigma^{\mu\nu}}, \tag{1}$$

with

$$\Sigma^{\mu\nu} \equiv \frac{1}{4i}[\gamma^\mu, \gamma^\nu], \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\ . \tag{2}$$

The explicit form for Dirac matrices in my conventions is

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\ .$$

We will often write

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_+^\mu \\ \sigma_-^\mu & 0 \end{pmatrix}\ ,$$

where $\sigma_\pm^\mu = (I, \pm\sigma_i)$.

1. Show that the $\Sigma_{\mu\nu}$ satisfy the Lorentz algebra. You can do in a generic manner, or even better, by separating the rotations and boosts, and then check the usual relations among rotations and boosts. If you get stuck, you can do it in the two-component form discussed below.

2. Introduce Left and Right chiral spinors

$$\Psi_{L,R} \equiv \frac{1 \pm \gamma_5}{2}\Psi \equiv L\,(R)\,\Psi\ , \tag{3}$$

$$\gamma_5 = -i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3\ ,$$

or

$$\gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\ .$$

Using Eq. (1), show that

$$u_{L(R)} \to e^{i\vec{\sigma}/2(\vec{\theta} \pm i\vec{\phi})} u_{L(R)}\ , \tag{4}$$

where

$$\Psi_L = \begin{pmatrix} u_L \\ 0 \end{pmatrix}, \quad \Psi_R = \begin{pmatrix} 0 \\ u_R \end{pmatrix}\ .$$

Clearly $\vec{\theta}$ represent rotations, with the usual three Euler angles, while $\vec{\phi}$ denotes boosts.

3. Take a boost in the z-direction $\vec{\phi} = \phi\hat{z}$ and find an expression for ϕ as a function of velocity.

4. Define the charge-conjugation transformation

$$\Psi^c \equiv C\overline{\Psi}^T \ , \tag{5}$$

with

$$C^T\gamma^\mu C = -\gamma_\mu^T, \ C^T = C^\dagger = C^{-1} = -C \ .$$

An explicit choice is $C = i\gamma_2\gamma_0$.

Show that

$$\Psi^c \to S\Psi^c \ , \tag{6}$$

when $\Psi \to S\Psi$. In other words, Ψ^c transforms the same way as Ψ, i.e. it is also a proper spinor. The only difference between Ψ^c and Ψ lies in their charges being opposite when they get coupled to a gauge field such as the photon.

5. Take

$$\Psi = \Psi_L = \begin{pmatrix} u_L \\ 0 \end{pmatrix} \ .$$

Compute Ψ^c. What is its chirality ?

6. What happens to u_L and u_R under parity ? By definition, under parity transformation

$$\Psi \to \gamma_0\Psi \ .$$

Problem 2

Now take a four-component Majorana spinor Ψ_M, defined by

$$\Psi_M = \Psi_M^c \ .$$

1. Show that Ψ_M can be written as

$$\Psi_M = \begin{pmatrix} u_L \\ -i\sigma_2 u_L^* \end{pmatrix} \ .$$

2. Consider the free Majorana Lagrangian

$$\mathcal{L} = i\bar{\Psi}_M\gamma^\mu\partial_\mu\Psi_M - m\bar{\Psi}_M\Psi_M \ . \tag{7}$$

Show that it is equivalent to

$$\mathcal{L} = iu_L^\dagger\sigma_-^\mu\partial_\mu u_L - \frac{1}{2}m_M\left(u_L^T\, i\sigma_2 u_L + \text{h.c.}\right) \ . \tag{8}$$

The mass term m_M is called Majorana mass term since it leads to a Majorana spinor.

3. Can a Majorana mass term conserve global continuous symmetry such as charge conjugation?

4. Show that the following vector current vanishes

$$\bar{\Psi}_M\gamma^\mu\Psi_M \ .$$

Can you explain why?

Hint: Can there be invariance under $\Psi_M \to e^{-i\alpha}\Psi_M$*?*

Problem 3

Take an arbitrary SU(N) group with element $U = \exp(i\theta^a T^a)$. Hereafter, the sum over repeated indices is assumed. The generators T^a in the fundamental representation (say spin 1/2 in SU(2)) define the corresponding Lie algebra

$$[T^a, T^b] = if^{abc}T^c \ ,$$

and

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab} \ .$$

1. Let us start with a generalized Dirac Lagrangian,

$$\mathcal{L}_D = i\bar{\psi}\gamma^\mu D_\mu \psi \ ,$$

where $D_\mu = \partial_\mu - ig\mathbf{A}_\mu$ and $\mathbf{A}_\mu \equiv A^a_\mu T^a$. Prove that this Lagrangian is invariant under the Yang-Mills transformation

$$\psi \to U\psi, \qquad \mathbf{A}_\mu \to U\mathbf{A}_\mu U^\dagger + (i/g)U\partial_\mu U^\dagger \ ,$$

where $U(x) = \exp[i\theta^a(x)T^a]$ is space-time dependent.

2. Let us define the Yang-Mills field strength

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - ig[\mathbf{A}_\mu, \mathbf{A}_\nu] \ ,$$

where $\mathbf{F}_{\mu\nu} \equiv F^a_{\mu\nu}T^a$.
Write the explicit expression for $F^a_{\mu\nu}$ as a function of A^a_μ.

Show that $\mathbf{F}_{\mu\nu}$ transforms as the so-called adjoint representation

$$\mathbf{F}_{\mu\nu} \to U\mathbf{F}_{\mu\nu}U^{-1} \ ,$$

and the Yang-Mills Lagrangian can be written as (meaning that it is gauge invariant)

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\mathrm{Tr}\left(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right) \ .$$

3. Now we go to the component form. Write down the transformation properties for A^a_μ and $F^a_{\mu\nu}$ in the case of infinitesimal $\theta^a(x)$.

4. Consider the complete Dirac and Yang-Mills Lagrangian

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_{\mathrm{YM}}$$

which, as you showed, is invariant under the YM transformation.

$$\psi \to U\psi, \qquad \mathbf{A}_\mu \to U\mathbf{A}_\mu U^\dagger + (i/g)U\partial_\mu U^\dagger \ .$$

Take constant θ_a, i.e. the global SU(2) transformation and find the conserved current using the Noether theorem.

5. Write down the equations of motion for both the fermion and gauge boson fields.

6. Any representation that transforms as the gauge bosons is called the adjoint representation. If we speak of scalars or fermions - or we just concentrate on the constant θ_a (global transformations), the adjoint transforms as

$$\mathbf{A} \to U\mathbf{A}U^\dagger \ ,$$

where $\mathbf{A} \equiv A^a T^a$.

In the case of SU(2), where $f_{abc} = \epsilon_{abc}$, show that the adjoint transforms as a vector of SO(3)

$$A \to OA \ ,$$

with

$$O = \exp[i\theta^a L_a], \quad (L_a)_{ij} = -i\epsilon_{ijk} \ ,$$

and

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \ .$$