Introduction to Physics of Neutrinos

Solutions to Homework 1

Problem 1

① $(4i)^2[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = [\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu, \gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho]$

$$= [\gamma_\mu\gamma_\nu, \gamma_\rho\gamma_\sigma] - [\gamma_\mu\gamma_\nu, \gamma_\sigma\gamma_\rho] - [\gamma_\nu\gamma_\mu, \gamma_\rho\gamma_\sigma]$$

$$+ [\gamma_\nu\gamma_\mu, \gamma_\sigma\gamma_\rho] = \gamma_\rho[\gamma_\mu\gamma_\nu, \gamma_\sigma] + [\gamma_\mu\gamma_\nu, \gamma_\rho]\gamma_\sigma$$

$$- \gamma_\sigma[\gamma_\mu\gamma_\nu, \gamma_\rho] - [\gamma_\mu\gamma_\nu, \gamma_\sigma]\gamma_\rho - \gamma_\rho[\gamma_\nu\gamma_\mu, \gamma_\sigma]$$

$$- [\gamma_\nu\gamma_\mu, \gamma_\rho]\gamma_\sigma + \gamma_\sigma[\gamma_\nu\gamma_\mu, \gamma_\rho] + [\gamma_\nu\gamma_\mu, \gamma_\sigma]\gamma_\rho$$

using $[\alpha\beta, \gamma] = \alpha\{\beta, \gamma\} - \{\alpha, \gamma\}\beta$, the above becomes:

$$(4i)^2[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = \gamma_\rho\big(\gamma_\mu\{\gamma_\nu, \gamma_\sigma\} - \{\gamma_\mu, \gamma_\sigma\}\gamma_\nu - \gamma_\nu\{\gamma_\mu, \gamma_\sigma\}$$

$$+ \{\gamma_\nu, \gamma_\sigma\}\gamma_\mu\big) - \big(\gamma_\mu\{\gamma_\nu, \gamma_\sigma\} - \{\gamma_\mu, \gamma_\sigma\}\gamma_\nu - \gamma_\nu\{\gamma_\mu, \gamma_\sigma\}$$

$$+ \{\gamma_\nu, \gamma_\sigma\}\gamma_\mu\big)\gamma_\rho - \gamma_\sigma\big(\gamma_\mu\{\gamma_\nu, \gamma_\rho\} - \{\gamma_\mu, \gamma_\rho\}\gamma_\nu - \gamma_\nu\{\gamma_\mu, \gamma_\rho\}$$

$$+ \{\gamma_\nu, \gamma_\rho\}\gamma_\mu\big) + \big(\gamma_\mu\{\gamma_\nu, \gamma_\rho\} - \{\gamma_\mu, \gamma_\rho\}\gamma_\nu - \gamma_\nu\{\gamma_\mu, \gamma_\rho\}$$

$$+ \{\gamma_\nu, \gamma_\rho\}\gamma_\mu\big) = 4\big([\gamma_\rho, \gamma_\mu]g_{\nu\sigma} - [\gamma_\rho, \gamma_\nu]g_{\mu\sigma}$$

$$- [\gamma_\sigma, \gamma_\mu]g_{\rho\nu} + [\gamma_\sigma, \gamma_\nu]g_{\mu\rho}\big)$$

$$\to [\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = i\big(\Sigma_{\rho\nu}g_{\mu\sigma} + \Sigma_{\sigma\mu}g_{\rho\nu} - \Sigma_{\rho\mu}g_{\nu\sigma}$$

$$- \Sigma_{\sigma\nu}g_{\mu\rho}\big)$$

Lorentz algebra commutation relations.

② ⊛ $\Sigma^{0i} = \frac{1}{4i}[\gamma^0,\gamma^i] = \frac{1}{4i}\left[\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\right]$

$$= \frac{1}{4i}\left\{\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} - \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}\right\}$$

$$= \frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} = \frac{i}{2}\sigma^i \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

⊛ $\Sigma^{ij} = \frac{1}{4i}[\gamma^i,\gamma^j] = \frac{1}{4i}\left[\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\right]$

$$= \frac{i}{2}\begin{pmatrix} [\sigma^i,\sigma^j] & 0 \\ 0 & [\sigma^i,\sigma^j] \end{pmatrix}$$

$$= -\frac{1}{2}\varepsilon_{ijk}\sigma_k \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$\rightarrow e^{i\theta_{\mu\nu}\Sigma^{\mu\nu}} = e^{i(\theta_{ij}\Sigma^{ij} + \theta_{0i}\Sigma^{0i})}$

$$= e^{i\left[-\frac{1}{2}\varepsilon_{ijk}\theta_{ij}\sigma_k \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} + \frac{i}{2}\theta_{0i}\sigma^i \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\right]}$$

$$= e^{i\frac{\sigma_k}{2}\left[-\theta_k \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} + i\phi_k \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\right]},$$

with $\theta_k = -\varepsilon_{ijk}\theta_{ij}$ & $\phi_k = \theta_{0k}$.

Acting with the above on the spinor, you find the desired result:

$$u_{L,R} = e^{\frac{i\vec{\sigma}}{2}(\vec{\theta} \pm i\vec{\varphi})}\, u_{L,R}.$$

③ A boost in the z-direction by an angle ϕ (ϕ is called "rapidity") is the "rotation"

$$t \to t' = t\,\mathrm{ch}\,\phi - z\,\mathrm{sh}\,\phi$$

$$x \to x' = x$$

$$y \to y' = y \qquad (*)$$

$$z \to z' = z\,\mathrm{ch}\,\phi - t\,\mathrm{sh}\,\phi\,.$$

At the same time,

$$t \to t' = \gamma(t - vz)$$

$$x \to x' = x \qquad (**)$$

$$y \to y' = y$$

$$z \to z' = \gamma(z - vt) \quad , \gamma^{-1} = \sqrt{1 - v^2}$$

Comparing $(*)$, $(**)$, we find:

$$\mathrm{ch}\,\phi = \gamma\,, \quad \mathrm{sh}\,\phi = \gamma v$$

$$\to \mathrm{th}\,\phi = v \quad \to \boxed{\phi = \mathrm{th}^{-1}(v)}$$

④ $\psi^c \equiv \varsigma \bar{\psi}^T = \varsigma(\psi^\dagger\gamma^0)^T = \varsigma\gamma^0\psi^*$

$$\psi^* \to \psi'^* = S'^*\psi^*$$

$$\to \psi'^c = \varsigma\gamma^0 S'^*\psi^*$$

Using $S'^\dagger = \gamma^0 S'^{-1}\gamma^0$, the above becomes:

$$\psi'^c = \varsigma S'^*\gamma^0\psi^* = \varsigma S'^*\bar{\psi}^T. \quad ⊛$$

Now we have to see how to exchange ς, S'^*. Notice the following:

$$\varsigma S'^* = \varsigma e^{-i\theta_{\mu\nu}\Sigma^*_{\mu\nu}} \simeq \varsigma(1 - i\theta_{\mu\nu}\Sigma^*_{\mu\nu} + \dots)$$

$$= \varsigma\left(1 - i\theta_{\mu\nu}\left(\frac{-1}{4i}\right)[\gamma_\mu^*, \gamma_\nu^*] + \dots\right)$$

Now $\varsigma^\dagger\gamma^\mu\varsigma = -(\gamma^\mu)^T = -(\gamma^\mu)^*$

$$\to \varsigma S'^* \simeq \varsigma\left(1 + i\theta_{\mu\nu}\left(\frac{1}{4i}\right)[\varsigma^\dagger\gamma_\mu\varsigma, \varsigma^\dagger\gamma_\nu\varsigma] + \dots\right)$$

$$= \varsigma + i\theta_{\mu\nu}\left(\frac{1}{4i}\right)[\gamma_\mu, \gamma_\nu]\varsigma + \dots$$

$$= (1 + i\theta_{\mu\nu}\Sigma_{\mu\nu} + \dots)\varsigma = e^{i\theta_{\mu\nu}\Sigma_{\mu\nu}}\varsigma$$

$$\to \varsigma S'^* = S'\varsigma.$$

Plugging the above into ⊛, we find

$\psi'^c = \varsigma S'^*\bar{\psi}^T = S'\varsigma\bar{\psi}^T = S'\psi^c$, as it should.

⑤ $\Psi = \begin{pmatrix} u_L \\ 0 \end{pmatrix}$, thus

$$\Psi_c = C\,\bar{\Psi}_M^T = i\gamma_2\gamma_0(\Psi_M^\dagger\gamma^0)^T$$

$$= i\gamma_2(\gamma_0)^2\,\Psi_M^* = i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} u_L^* \\ 0 \end{pmatrix}$$

$$\rightsquigarrow \Psi_c = \begin{pmatrix} 0 \\ -i\sigma^2 u_L^* \end{pmatrix}$$

$$\left(\frac{1+\gamma_5}{2}\right)\Psi_c = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -i\sigma^2 u_L^* \end{pmatrix} = 0$$

$$\left(\frac{1-\gamma_5}{2}\right)\Psi_c = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} 0 \\ -i\sigma^2 u_L^* \end{pmatrix} = \Psi^c$$

→ the fermion is right-handed.

⑥ $\Psi = \Psi_L + \Psi_R = \begin{pmatrix} u_L \\ u_R \end{pmatrix}$

$$\rightarrow \Psi \mapsto \Psi' = \gamma^0\Psi = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} u_L \\ u_R \end{pmatrix}$$

$$= \begin{pmatrix} u_R \\ u_L \end{pmatrix} \rightsquigarrow u_L \leftrightarrow u_R$$

flips chirality under the parity transformation

Problem 2

We have $\Psi = \Psi_L + \Psi_R = \begin{pmatrix} u_L \\ u_R \end{pmatrix}$

① $\Psi_M^c \equiv C \, \overline{\Psi}_M^T = C(\Psi_M^+ \gamma^0)^T$

$$= C(\gamma^0)^T \Psi_M^* = C\gamma^0 \Psi_M^*,$$

since $(\gamma^0)^T = \gamma^0$. Now $C = i\gamma_2\gamma_0$, thus

$\Psi_M^c = i\gamma_2(\gamma_0)^2 \Psi_M^* = i\gamma_2 \Psi_M^*,$

where we used $(\gamma_0)^2 = 1$.

Explicitely, $\gamma_2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}$,

therefore

$$\Psi_M^c = \begin{pmatrix} i\sigma^2 u_R^* \\ -i\sigma^2 u_L^* \end{pmatrix} \quad ⊛$$

For a Majorana spinor $\Psi_M = \Psi_M^c$, so from ⊛, we immediately find

$$\Psi_M = \begin{pmatrix} u_L \\ -i\sigma^2 u_L^* \end{pmatrix}.$$

$$② \quad \mathcal{L} = \underbrace{i\bar{\Psi}_M \gamma^\mu \partial_\mu \Psi_M}_{\mathcal{L}_1} - \underbrace{m_M \bar{\Psi}_M \Psi_M}_{\mathcal{L}_2}$$

Let's look at each term separately:

$$\circledast \quad \mathcal{L}_1 = i \Psi_M^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi_M$$

$$= i\,(u_L^\dagger,\ u_L^T i\sigma^2) \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_+^\mu \\ \sigma_-^\mu & 0 \end{pmatrix} \partial_\mu \begin{pmatrix} u_L \\ -i\sigma^2 u_L^* \end{pmatrix}$$

$$= i\,(u_L^T i\sigma^2,\ u_L^\dagger) \begin{pmatrix} \sigma_+^\mu (-i\sigma^2) \partial_\mu u_L^* \\ \sigma_-^\mu \partial_\mu u_L \end{pmatrix}$$

$$= i\left(u_L^T (i\sigma^2) \sigma_+^\mu (-i\sigma^2) u_L^* + u_L^\dagger \sigma_-^\mu \partial_\mu u_L\right)$$

$$= i\left[u_L^\dagger \sigma_-^\mu \partial_\mu u_L + (i\sigma^2 u_L^*)^\dagger \sigma_+^\mu \partial_\mu (i\sigma^2 u_L^*)\right)$$

One can explicitly show (show it!!!) that the second term is equal to the first.

So $\mathcal{L}_1 = 2i\,(u_L^\dagger \sigma_-^\mu \partial_\mu u_L)$

$$\circledast \quad \mathcal{L}_2 = m_M \bar{\Psi}_M \Psi_M = m \Psi_M^\dagger \gamma^0 \Psi_M$$

$$= m_M\left(u_L^T (i\sigma^2) u_L + u_L^\dagger (-i\sigma^2) u_L^*\right)$$

Putting everything together:

$$\mathcal{L} = 2\left[i(u_L^\dagger \sigma_+^\mu \partial_\mu u_L - \frac{m_M}{2}\left(u_L^T (i\sigma^2) u_L + h.c.\right)\right]$$

(3) Charge conjugation:

$$\Psi_\mu \to \Psi_\mu^c \quad , \text{but} \quad \Psi_\mu^c = \Psi_\mu$$

$$\to \bar{\Psi}_\mu = \Psi_\mu^+ \gamma^0 = (\Psi_\mu^c)^+ \gamma^0$$

$$= \Psi_\mu^+ \gamma^0 = \bar{\Psi}_\mu$$

$\to$ invariant

(4) The current must be zero, because the theory is not invariant under the global $U(1)$

$$\Psi_\mu \to \Psi_\mu' = e^{-i\alpha} \Psi_\mu .$$

The symmetry is killed by virtue of the Majorana constraint

$$\Psi_\mu = \Psi_\mu^c .$$

One can explicitly show (show it!!!) that

$\bar{\Psi}_\mu \gamma^\mu \Psi_\mu = - \bar{\Psi}_\mu \gamma^\mu \Psi_\mu \to$ the current is identically 0.

① $\mathcal{L}_D = i\bar{\Psi}\gamma^\mu \partial_\mu \Psi + g\bar{\Psi}\gamma^\mu A_\mu \Psi$

$\rightarrow \mathcal{L}_D' = i\bar{\Psi}'\gamma^\mu \partial_\mu \Psi' + g\bar{\Psi}'\gamma^\mu A_\mu' \Psi'$

$= i\bar{\Psi}U^\dagger\gamma^\mu \partial_\mu (U\Psi) + g\bar{\Psi}U^\dagger\gamma^\mu (UA_\mu U^\dagger + i/g\, U\partial_\mu U^\dagger) U\Psi$

$= i\bar{\Psi}\gamma^\mu\partial_\mu\Psi + i\bar{\Psi}U^\dagger\gamma^\mu\partial_\mu U\Psi + g\bar{\Psi}\gamma^\mu A_\mu\Psi + i\bar{\Psi}\gamma^\mu\partial_\mu U^\dagger U\Psi$

using $\quad U^\dagger U = 1 \rightarrow \partial_\mu U^\dagger U = -U^\dagger\partial_\mu U$

$\rightarrow \mathcal{L}_D' = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi + g\bar{\Psi}\gamma^\mu A_\mu\Psi = i\bar{\Psi}\gamma^\mu D_\mu\Psi = \mathcal{L}_D$

$\rightarrow$ SU(N) invariant Lagrangian.

② $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ ⊗

$$\Rightarrow F^c_{\mu\nu} T^c = \partial_\mu A^c_\nu T^c - \partial_\nu A^c_\mu T^c - ig[A^a_\mu T^a, A^b_\nu T^b]$$

$$= \partial_\mu A^c_\nu T^c - \partial_\nu A^c_\mu T^c - ig A^a_\mu A^b_\nu [T^a, T^b]$$

$$= \partial_\mu A^c_\nu T^c - \partial_\nu A^c_\mu T^c + g f^{abc} A^a_\mu A^b_\nu T^c$$

$$\boxed{\Rightarrow F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g f^{abc} A^a_\mu A^b_\nu}$$

We know that $A_\mu \to A'_\mu = U A_\mu U^+ + i/g\, U \partial_\mu U^+$

From ⊗ $\Rightarrow$

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu - ig[A'_\mu, A'_\nu]$$

$$= \partial_\mu (U A_\nu U^+ + i/g\, U \partial_\nu U^+) - \partial_\nu (U A_\mu U^+ + i/g\, U \partial_\mu U^+)$$

$$- ig[U A_\mu U^+ + i/g\, U \partial_\mu U^+, U A_\nu U^+ + i/g\, U \partial_\nu U^+]$$

$$= \partial_\mu U A_\nu U^+ + \underline{U \partial_\mu A_\nu U^+} + U A_\nu \partial_\mu U^+$$

$$+ i/g\, \partial_\mu U \partial_\nu U^+ + i/g\, U \cancel{\partial_\mu \partial_\nu} U^+ - \partial_\nu U A_\mu U^+$$

$$- \underline{U \partial_\nu A_\mu U^+} - U A_\mu \partial_\nu U^+ - i/g\, \partial_\nu U \partial_\mu U^+$$

$$- i/g\, U \cancel{\partial_\nu \partial_\mu} U^+ - \underline{ig[U A_\mu U^+, U A_\nu U^+]}$$

$$+ [U A_\mu U^+, U \partial_\nu U^+] + [U \partial_\mu U^+, U A_\nu U^+]$$

$$+\frac{i}{g}[U\partial_\mu U^\dagger, U\partial_\nu U^\dagger]$$

$$= U F_{\mu\nu} U^\dagger + \partial_\mu U A_\nu U^\dagger + \cancel{U A_\nu \partial_\mu U^\dagger}$$
$$+ \frac{i}{g}\partial_\mu U \partial_\nu U^\dagger - \frac{i}{g}\partial_\nu U \partial_\mu U^\dagger - \partial_\nu U A_\mu U^\dagger$$
$$- \cancel{U A_\mu \partial_\nu U^\dagger} + \cancel{U A_\mu \partial_\nu U^\dagger}$$
$$- U\partial_\nu U^\dagger U A_\mu U^\dagger + U\partial_\mu U^\dagger U A_\nu U^\dagger$$
$$- \cancel{U A_\nu \partial_\mu U^\dagger} + \frac{i}{g} U\partial_\mu U^\dagger U \partial_\nu U^\dagger$$
$$- \frac{i}{g} U\partial_\nu U^\dagger U \partial_\mu U^\dagger$$

$$= U F_{\mu\nu} U^\dagger + \partial_\mu U A_\nu U^\dagger + U\partial_\mu U^\dagger U A_\nu U^\dagger$$
$$- \partial_\nu U A_\mu U^\dagger - U\partial_\nu U^\dagger U A_\mu U^\dagger$$
$$+ \frac{i}{g}\partial_\mu U \partial_\nu U^\dagger + \frac{i}{g} U\partial_\mu U^\dagger U\partial_\nu U^\dagger$$
$$- \frac{i}{g}\partial_\nu U \partial_\mu U^\dagger - \frac{i}{g} U\partial_\nu U^\dagger U \partial_\mu U^\dagger$$

Now we use: $U^\dagger U = 1 \rightarrow \partial_\mu U^\dagger U = -U^\dagger \partial_\mu U$

$$\rightarrow F_{\mu\nu}' = U F_{\mu\nu} U^\dagger + \partial_\mu U A_\nu U^\dagger - U U^\dagger \partial_\mu U A_\nu U^\dagger$$
$$- \partial_\nu U A_\mu U^\dagger + U U^\dagger \partial_\nu U A_\mu U^\dagger$$
$$+ \frac{i}{g}\partial_\mu U \partial_\nu U^\dagger - \frac{i}{g} U U^\dagger \partial_\mu U \partial_\nu U^\dagger$$
$$- \frac{i}{g}\partial_\nu U \partial_\mu U^\dagger + \frac{i}{g} U U^\dagger \partial_\nu U \partial_\mu U^\dagger = U F_{\mu\nu} U^\dagger.$$

$\rightarrow F_{\mu\nu}$ transforms in the adjoint of $SU(N)$.

We notice that

$$\mathcal{L}_{YM} = -\frac{1}{2} Tr(F_{\mu\nu} F^{\mu\nu})$$

$$\rightarrow \mathcal{L}'_{YM} = -\frac{1}{2} Tr(F'_{\mu\nu} F'^{\mu\nu}) = -\frac{1}{2} Tr(\mathcal{U} F_{\mu\nu} \mathcal{U}^{+} \mathcal{U} F^{\mu\nu} \mathcal{U}^{+})$$

$$= -\frac{1}{2} Tr(F_{\mu\nu} F^{\mu\nu}) = \mathcal{L}_{YM}$$, due to the cyclic property of the trace.

③ $A_\mu \rightarrow A'_\mu = \mathcal{U} A_\mu \mathcal{U}^{+} + i/g\, \mathcal{U} \partial_\mu \mathcal{U}^{+}$

$$\simeq (1 + i\theta_a T_a) A_\mu (1 - i\theta_b T_b) + \frac{1}{g} (1 + i\theta_a T_a) \partial_\mu \theta_b T_b$$

$$= (A_\mu + i\theta_a T_a A_\mu)(1 - i\theta_b T_b) + \frac{1}{g} \partial_\mu \theta_c T_c + \ldots$$

$$= A_\mu - i A_\mu \theta_a T_a + i\theta_a T_a A_\mu + \frac{1}{g} \partial_\mu \theta_c T_c$$

$$= A_\mu + i\theta_a A_\mu^b [T_a, T_b] + \frac{1}{g} \partial_\mu \theta_c T_c$$

$$= A_\mu^c T^c - f^{abc} \theta^a A_\mu^b T_c + \frac{1}{g} \partial_\mu \theta_c T_c$$

$$\Rightarrow A_\mu^{c\prime} = A_\mu^c - f^{abc} \theta^a A_\mu^b + \frac{1}{g} \partial_\mu \theta_c .$$

We also easily find that

$$F'^{c}_{\mu\nu} = F^c_{\mu\nu} - f^{abc} \theta^a F^b_{\mu\nu} .$$

④ $A_\mu \to A'_\mu = \mathcal{U} A_\mu \mathcal{U}^+ = (1 + i\theta_a T_a) A_\mu (1 - i\theta_b T_b)$

$$= A_\mu + i\theta_a [T_a, A_\mu]$$

$\rightsquigarrow \delta A_\mu = -f^{abc}\theta^a A_\mu^b T^c$

$\rightsquigarrow \delta A_\mu^a = -f^{ija}\theta^i A_\mu^j$

$$j_\mu^a \supset \frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu^b} f^{abc} A_\nu^c = -\frac{1}{4}\frac{\delta}{\delta \partial_\mu A_\nu^b}\left(F_{\lambda\sigma}^i F_{\lambda\sigma}^i\right) f^{abc} A_\nu^c$$

$$= -\frac{1}{2} F_{\lambda\sigma}^i \frac{\delta F_{\lambda\sigma}^i}{\delta \partial_\mu A_\nu^b} f^{abc} A_\nu^c$$

$$= -\frac{1}{2} F_{\lambda\sigma}^i \left(\delta_{\mu\lambda}\delta_{\nu\sigma}\delta^{bi} - \delta_{\nu\lambda}\delta_{\mu\sigma}\delta^{bi}\right) f^{abc} A_\nu^c$$

$\rightsquigarrow j_\mu^a \supset -f^{abc} F_{\mu\nu}^b A_\nu^c$

In addition to the above, we also have

$\psi \to \psi' = \mathcal{U}\psi \simeq (1 + i\theta_a T_a)\psi$

$\rightsquigarrow \delta\psi = i\theta_a T_a \psi$

$\mathcal{L}_D = i\left(\bar{\psi}\not{\partial}\psi - ig\bar{\psi}\not{A}\psi\right)$

$\frac{\delta \mathcal{L}_D}{\delta \partial_\mu \psi}\delta\psi = -\bar{\psi}\gamma^\mu T^a \psi$

Putting everything together

$$J^a_\mu = -f^{abc} F^b_{\mu\nu} A^c_\nu - \bar{\psi}\gamma^\mu T^a \psi .$$

(5) The equations of motion are obtained by varying the action w.r.t. the fields.

We find:

$$i\gamma^\mu D_\mu \Psi = 0 ,$$

$$i D_\mu \bar{\Psi} \gamma^\mu = 0 ,$$

$$D^\nu F^a_{\mu\nu} = g \bar{\Psi} \gamma^\mu T^a \Psi .$$

(6) $A \to A' = UAU^\dagger = e^{i\theta_a T_a} A \, e^{-i\theta_b T_b}$

Baker - Campbell - Hausdorff formula

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X,[X,Y]] + \cdots$$

with $X = i\theta_a T_a$, $Y = A$, we find

$$A' = A + [i\theta_a T_a, A] + \frac{1}{2!}[i\theta_a T_a, [i\theta_b T_b, A]] + \cdots$$

$$= A + i\theta_a A_b [T_a, T_b] - \frac{1}{2}\theta_a\theta_b A_c [T_a, [T_b, T_c]] + \cdots$$

$$= A - \theta_a A_b \varepsilon_{abc} T_c - \frac{i}{2}\theta_a\theta_b A_c \varepsilon_{bcd}[T_a, T_d] + \cdots$$

$$= A - \theta_a A_b \varepsilon_{abc} T_c + \frac{1}{2}\theta_a\theta_b A_c \varepsilon_{bcd}\varepsilon_{ade} T_e$$

$$= A - \varepsilon_{abc}\theta_a A_b T_c - \frac{1}{2}\theta_a\theta_b A_c(\delta_{ab}\delta_{ce} - \delta_{eb}\delta_{ca}) T_e$$

$$= A_c T_c - \varepsilon_{abc}\theta_a A_b T_c - \frac{1}{2}(\theta^2 A_c T_c - \theta_a A_a \theta_c T_c)$$

$$\to A_c = A_c - \varepsilon_{abc}\theta_a A_b - \frac{1}{2}(\theta^2\delta_{bc} - \theta_b\theta_c) A_b$$

$$A_c = A_b\left(\delta_{bc} - \varepsilon_{abc}\theta_a - \frac{1}{2}(\theta^2\delta_{bc} - \theta_b\theta_c) + \cdots\right) \circledast$$

$$A_c = O_{cb} A_b \simeq \left(1 + i\theta_a L_a - \frac{1}{2}(\theta_a L_a)^2 + \cdots\right)_{cb} A_b$$

$$\simeq A_b\left(\delta_{bc} - \varepsilon_{bca}\theta_a + \frac{1}{2}(\theta^2\delta_{bc} - \theta_b\theta_c) + \cdots\right)$$

$$= \circledast$$

$\rightsquigarrow$ The adjoint of $SU(2)$ transforms as a vector of $SO(3)$. Do you know why?