Ludwig-Maximilians-Universität München

QCD and Standard Model

Exam Solution

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Guidelines :

- The exam consists of 7 problems.
- The duration of the exam is 4 hours.
- Please write your name or matriculation number on every sheet that you hand in.
- Your answers should be comprehensible and readable.

GOOD LUCK!

Exercise 1	9 P
Exercise 2	23 P
Exercise 3	13 P
Exercise 4	9 P
Exercise 5	20 P
Exercise 6	16 P
Exercise 7	10 P

Total	100 P

Problem 1 (9 points)

Consider the following Lagrangian of a real scalar field Φ in 4 spacetime dimensions

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \frac{m^2}{2}\Phi^2 - \frac{\lambda}{4}\Phi^4 ,$$

with $m^2, \lambda > 0$.

a) (2pts) What is the symmetry of the above Lagrangian ?
Answer : Since the scalar is real, the Lagrangian is invariant under

$$\Phi \to -\Phi .$$

b) (4pts) Minimize the potential and determine the ground state of the system. Is the symmetry broken spontaneously ? (1pt for each minimum + 2pts for SSB)
Answer : The potential is minimized by

$$\langle \Phi \rangle = \pm\sqrt{\frac{m^2}{\lambda}} \equiv \pm v .$$

Since the VEV of Φ is non-zero, the symmetry is broken spontaneously.

c) (3pts) How many Goldstone bosons are in the spectrum ? Justify your answer. (1pt for Goldstones number + 2pts for justification)
Answer : Since no continuous symmetry is broken, there are no Goldstone bosons.

Problem 2 (23 points)

Consider a theory invariant under a local SU(2) symmetry with a scalar field in the adjoint representation of the group, i.e. $\phi = \sum_{i=1}^{3} \phi^i T^i$, where the ϕ^i's are real and T^i's the Hermitian generators of SU(2).

a) (5pts)Write down the most general renormalizable SU(2)- and Lorentz-invariant Lagrangian in four spacetime dimensions.(1pt for each term+1pt for covariant derivative)
Answer : The most general gauge invariant & renormalizable Lagrangian reads

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \mathrm{Tr}[(D_\mu\phi)^\dagger D^\mu\phi] + m^2\mathrm{Tr}(\phi^2) - \lambda(\mathrm{Tr}(\phi)^2)^2 , \quad (1)$$

where the covariant derivative for the field in the adjoint representation reads

$$D_\mu\phi = \partial_\mu\phi - ig[A_\mu, \phi] \ , \tag{2}$$

and as usual the square brackets $[\cdot,\cdot]$ stand for the commutator.

b) (4pts)Arrange the potential such that the vacuum expectation value (vev) of the scalar field is non-zero. Find the vev by explicitly minimizing the potential. (2pts for coefficients+2pts for vev)
<u>Answer :</u> The potential that allows for spontaneous symmetry breaking reads

$$V(\phi) = -m^2\mathrm{Tr}(\phi^2) + \lambda(\mathrm{Tr}(\phi^2))^2 \ , \tag{3}$$

with $m^2, \lambda > 0$. Taking the derivative w.r.t. ϕ, we obtain

$$V'(\phi) = -2m^2\phi + 4\lambda\mathrm{Tr}(\phi^2)\phi \ . \tag{4}$$

We now require that for $\phi = \phi_0 \neq 0$ the above vanishes. This is the case for

$$\mathrm{Tr}(\phi_0^2) = \frac{m^2}{2\lambda} \ . \tag{5}$$

Since the field lives in the adjoint of the group, we have

$$\phi = \sum_{i=1}^{3}\phi^i T^i = \frac{1}{2}\begin{pmatrix} \phi^3 & \phi^1 - i\phi^2 \\ \phi^1 + i\phi^2 & -\phi^3 \end{pmatrix} \ , \tag{6}$$

and we immediately find

$$(\phi_0^1)^2 + (\phi_0^2)^2 + (\phi_0^3)^2 = \frac{m^2}{\lambda} \ . \tag{7}$$

Without loss of generality (due to the SU(2) symmetry), we may choose $\phi_0^1 = \phi_0^2 = 0$ and $\phi_0^3 = \frac{m}{\sqrt{\lambda}}$, meaning that

$$\phi_0 = \frac{v}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{8}$$

with $v = \frac{m}{\sqrt{\lambda}}$. To verify that this saddle point corresponds to a (local) minimum, we take the second derivative of the potential

$$V''(\phi)\Big|_{\phi=\phi_0} = 2m^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{9}$$

c) (5pts)What is the unbroken symmetry group? How many gauge bosons acquire a mass and how many remain massless? Justify your answer. (3pts for U(1)+2pts for spectrum)
Answer : To find the unbroken symmetry group, we have to study the action of the generators on the vev ϕ_0. Since the field lives in the adjoint, we have to consider its commutators with the generators :

$$\begin{aligned}
[\phi_0, T^1] &= v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \\
[\phi_0, T^2] &= -i\, v \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \\
[\phi_0, T^3] &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .
\end{aligned} \tag{10}$$

The unbroken symmetry group is therefore U(1). Out of the 3 gauge bosons, two will get a mass and one will be massless.

d) (6pts) Determine the masses of the gauge bosons.(3pts for formula+3pts for result)
Answer : The masses of the gauge bosons are generated via the covariant derivative of the scalar field and more specifically from the term

$$\mathrm{Tr}([A_\mu, \phi_0][A_\mu, \phi_0]) = \frac{g^2 v^2}{2}((A^1_\mu)^2 + (A^2_\mu)^2) , \tag{11}$$

meaning that the masses of the physical gauge bosons $W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \mp A^2_\mu)$ are equal to gv, while the mass of A^3_μ is zero. The above is of course in agreement with our expectation that only two of the gauge bosons will acquire a mass, while the third will remain massless.

e) (3pts) Can this model alone describe the gauge electroweak interactions? Justify your answer.(1pt for answer+2 for justification)
Answer : This model alone cannot describe the gauge electroweak sector of the Standard Model, because it cannot account for the Z boson.

Problem 3 (13 points)

a) (4pts) Add a right-handed neutrino to the Lagrangian of the SM, which only has a Majorana mass term. Write down the corresponding Lagrangian. Does this particle conserve lepton number? Verify your statement by an explicit calculation.(1pt for kinetic + 1pt for mass +

1pt for lepton number + 1pt for justification)

Answer :

$$\mathcal{L}_{\text{add}} = i\bar{\nu}_R\gamma^\mu\partial_\mu\nu_R - \frac{1}{2}m_M\left(\nu_R^T C\nu_R + \text{h.c.}\right),$$

where ν_R is the right-handed neutrino and $C = i\gamma^2\gamma^0$ is the charge conjugation matrix.

Since ν_R is a lepton, it transforms under $U(1)_{\text{lepton}}$ as

$$\nu_R \to e^{i\alpha}\nu_R.$$

But then, the above mass term is not invariant :

$$\nu_R^T C\nu_R \to e^{2i\alpha}\nu_R^T C\nu_R,$$

so lepton number is not conserved.

b) (3pts) Is a Majorana mass term invariant under parity transformations ? Verify your statement by an explicit calculation. (1pt for answer+2pts for computation)

Hint : Do not forget the phase.

Answer : A Majorana fermion transforms under parity as

$$\nu_R(x) \xrightarrow{P} i\gamma^0\nu_R(x'),$$

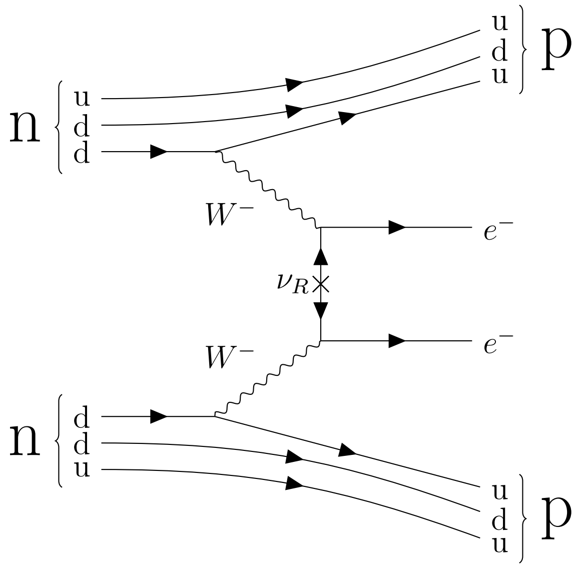
where $x' = (t, -\vec{x})$. So the mass term transforms as

$$\nu_R^T(x)C\nu_R(x) \to (i)^2\nu_R^T(x')\gamma^0\underbrace{C\gamma^0}_{-\gamma^0 C}\nu_R(x') = \nu_R^T(x')C\nu_R(x').$$

Hence the action (after performing $x' \to x$) is invariant.

c) (6pts) Draw a Feynman diagram for the process $n + n \to p + p + e^- + e^-$. Can this process happen within the SM (including the above modification) ? (4pts for diagram + 2pts for answering)

Answer :

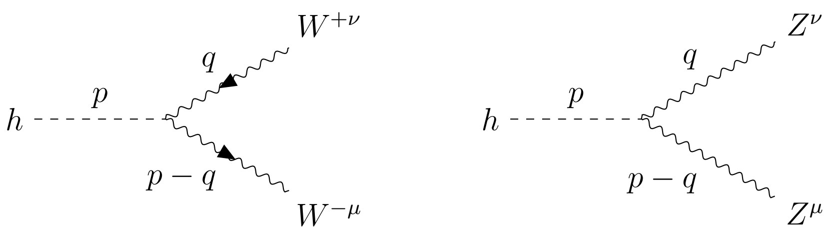


Yes, the process can happen, because ν_R is a Majorana particle and hence lepton number is not conserved.

Problem 4 (9 points)

a) (4pts) Draw two tree-level Feynman diagrams that describe the decay of the Higgs particle to the W and Z gauge bosons (one each).(2pts for each diagram)

Answer :



b) (3pts)Are these processes kinematically allowed ?

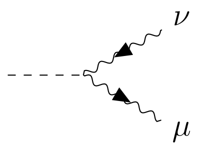
Answer : No, because $m_h \simeq 126\,\text{GeV}$, $m_W \simeq 80\,\text{GeV}$ and $m_Z \simeq 91\,\text{GeV}$, so the Higgs cannot decay into any of the two on-shell.

c) (2pts) Write down the interaction vertex governing the Higgs-W^+W^- coupling.

Answer : If we write the Higgs doublet as $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix}$ (in unitary gauge), the kinetic term $(D_\mu H)^\dagger D^\mu H$ in the SM Lagrangian contains the term

$$\frac{2m_W^2}{v} h W_\mu^+ W^{-\mu}.$$

From this we can read off the vertex



$$= i\frac{2m_W^2}{v} g^{\mu\nu}.$$

Problem 5 (20 points)

Let us focus on the fermionic sector of the SM (without right-handed neutrinos).

a) (10pts)Set all the Yukawa couplings to zero. What is the global symmetry group in this case (apart from the gauged $SU(2)\times U(1)$) ? (5pts for result (2.5 if only subgroups or less symmetries+5pts for justification))

Answer : Setting all Yukawa coupling to zero, the fermionic sector only consists of kinetic terms, i.e.,

$$\begin{aligned}\mathcal{L}_\mathrm{F} =&(Q_L)^a\ i\gamma^\mu D_\mu (Q_L)_a + (u_R)^a\ i\gamma^\mu D_\mu (u_R)_a + (d_R)^a\ i\gamma^\mu D_\mu (d_R)_a \\ &+ (L_L)^a\ i\gamma^\mu D_\mu (L_L)_a + (e_R)^a\ i\gamma^\mu D_\mu (e_R)_a\ ,\end{aligned}$$

where $a = 1, 2, 3$ is a generation and D_μ is the standard covariant derivative for the corresponding fermionic field. In this form we can see that the Lagrangian is invariant under

$$\begin{aligned}(Q_L)^a &\longrightarrow (U_{Q_L})^a_b (Q_L)^b \\ (u_R)^a &\longrightarrow (U_{u_R})^a_b (u_R)^b \\ (d_R)^a &\longrightarrow (U_{d_R})^a_b (d_R)^b \\ (L_L)^a &\longrightarrow (U_{L_L})^a_b (L_L)^b \\ (e_R)^a &\longrightarrow (U_{e_R})^a_b (e_R)^b\ ,\end{aligned}$$

where $U_{Q_L}, U_{u_R}, U_{d_R}, U_{L_L}, U_{e_R} \in \mathrm{U}(3)$ and independent of each other. Thus, the global Flavor symmetry is

$$G_\mathrm{F} = \mathrm{U}(3)_{Q_L} \times \mathrm{U}(3)_{u_R} \times \mathrm{U}(3)_{d_R} \times \mathrm{U}(3)_{L_L} \times \mathrm{U}(3)_{e_R}$$

b) (10pts) Assume that the Yukawa couplings are non-zero diagonal matrices. What is the global symmetry group in this case ? (5pts for result (2.5 if only subgroups or less symmetries+5pts for justification))

Answer : The Yukawa terms are

$$\begin{aligned}\mathcal{L}_{\mathrm{Y}} = &- H(y^{(d)})_{ab}(\bar{Q}_L)^a(d_R)^b - \tilde{H}(y^{(u)})_{ab}(\bar{Q}_L)^a(u_R)^b \\ &- \tilde{H}(y^{(e)})_{ab}(\bar{L}_L)^a(e_R)^b + h.c. \,.\end{aligned}$$

We want to stress that the Yukawa matrices are taken to be non-zero diagonal matrices. In this case, the flavor symmetry is explicitly broken down to the vectorlike subgroups

$$\mathrm{U}(1)_{Q^1} \times \mathrm{U}(1)_{Q^2} \times \mathrm{U}(1)_{Q^3} \times \mathrm{U}(1)_{e^1} \times \mathrm{U}(1)_{e^2} \times \mathrm{U}(1)_{e^3} \tag{12}$$

i.e,

$$\begin{aligned}(Q_L)^a &\longrightarrow \exp(i\alpha^{(a)})(Q_L)^a \\ (u_R)^a &\longrightarrow \exp(i\alpha^{(a)})(u_R)^a \\ (d_R)^a &\longrightarrow \exp(i\alpha^{(a)})(d_R)^a \\ (L_L)^a &\longrightarrow \exp(i\beta^{(a)})(L_L)^a \\ (e_R)^a &\longrightarrow \exp(i\beta^{(a)})(e_R)^a \,,\end{aligned}$$

where $\exp(i\alpha^{(a)}) \in \mathrm{U}(1)_{Q^a}$ and $\exp(i\beta^{(a)}) \in \mathrm{U}(1)_{e^a}$. In the leptonic sector the U(1)s correspond to the conservation of electrons + electron neutrinos (electron number), myons + myon neutrinos (myon number), and tau + tau neutrinos (tau number). In the quark sector the U(1)s correspond to the conservation of up quarks + down quarks, charm quarks + strange quarks, and top quarks + bottom quarks.

If the Yukawa couplings were non-diagonal, the mixing would lead to a further breaking of the three quark U(1)s into the $\mathrm{U}(1)_{\mathrm{B}}$ subgroup, where $\alpha^1 = \alpha^2 = \alpha^3$, corresponding to the conservation of Baryon number.

Problem 6 (16 points)

Consider the following Lagrangian of a complex scalar field ϕ in 4 spacetime dimensions

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi - \lambda(\phi^*\phi)^2 + \mu^2(\phi^* + \phi)^2 \,.$$

Here $m^2, \lambda, \mu^2 > 0$ and $\mu \ll m$ are real parameters.

a) (4pts) Determine the ground state of the theory.
Answer : The potential reads

$$V = -m^2\phi^*\phi + \lambda(\phi^*\phi)^2 - \mu^2(\phi^* + \phi)^2 \ . \tag{13}$$

Let us introduce the real fields ϕ_1 and ϕ_2, such that

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \ . \tag{14}$$

The potential becomes

$$V = -\frac{m^2 + 4\mu^2}{2}\phi_1^2 - \frac{m^2}{2}\phi_2^2 + \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2 \ . \tag{15}$$

First we take the derivatives of the above w.r.t. ϕ_1 and ϕ_2 and we require that they vanish

$$\partial_{\phi_1} V\Big|_{\phi_1=\phi_1^0,\phi_2=\phi_2^0} = \partial_{\phi_2} V\Big|_{\phi_1=\phi_1^0,\phi_2=\phi_2^0} = 0 \ . \tag{16}$$

The above equations admit the following solutions

$$s_1 = \{\phi_1^0 = \phi_2^0 = 0\} \ , \tag{17}$$

$$s_2 = \left\{\phi_1^0 = 0, \phi_2^0 = \pm\frac{m}{\sqrt{\lambda}}\right\} , \tag{18}$$

$$s_3 = \left\{\phi_1^0 = \pm\sqrt{\frac{m^2 + 4\mu^2}{\lambda}}, \phi_2^0 = 0\right\} \ , \tag{19}$$

To determine which solution corresponds to a (local) minimum of the potential, we compute the "mass matrix" of the theory, i.e. the matrix of the second derivatives of the potential on top of the saddles s_i.
First consider s_1, for which

$$M(s_1) = \begin{pmatrix} \partial_{\phi_1,\phi_1} V & \partial_{\phi_1,\phi_2} V \\ \partial_{\phi_2,\phi_1} V & \partial_{\phi_2,\phi_2} V \end{pmatrix}\Big|_{s_1} = -\begin{pmatrix} m^2 & 0 \\ 0 & m^2 + 4\mu^2 \end{pmatrix} \ . \tag{20}$$

The eigenvalues of the above are negative, meaning that s_1 is not a minimum of the potential.
Turning to s_2, we obtain

$$M(s_2) = \begin{pmatrix} \partial_{\phi_1,\phi_1} V & \partial_{\phi_1,\phi_2} V \\ \partial_{\phi_2,\phi_1} V & \partial_{\phi_2,\phi_2} V \end{pmatrix}\Big|_{s_2} = 2\begin{pmatrix} -2\mu^2 & 0 \\ 0 & m^2 \end{pmatrix} \ . \tag{21}$$

Since $\mu^2 > 0$, s_2 does not minimize the potential.
Finally for s_3, we obtain

$$M(s_3) = \begin{pmatrix} \partial_{\phi_1,\phi_1} V & \partial_{\phi_1,\phi_2} V \\ \partial_{\phi_2,\phi_1} V & \partial_{\phi_2,\phi_2} V \end{pmatrix} \Big|_{s_3} = 2 \begin{pmatrix} m^2 + 4\mu^2 & 0 \\ 0 & 2\mu^2 \end{pmatrix} > 0 \ , \quad (22)$$

implying that s_3 corresponds to the vacuum of the theory.

b) (7pts) Find the masses of the particles when the Lagrangian is expanded around the vacuum. (4pts computation + 3pts for result)
Answer : The masses squared of the particles are the eigenvalues of the matrix $M(s_3)$, i.e.

$$m_1^2 = 4\mu^2 \ , \quad m_2^2 = 2(m^2 + 4\mu^2) \ . \quad (23)$$

c) (5pts) What happens in the limit $\mu \to 0$? Why? Explain. (3pts for U(1)+2pts for explanation (Goldstone))
Answer : If we take $\mu \to 0$, the theory acquires a global (continuous) U(1) symmetry. The ground state s_3 (or equivalently s_2) breaks this symmetry spontaneously, so there is one massless Goldstone boson in the spectrum. This is explicitly verified by setting $\mu = 0$ in the expressions for masses of the particles from point b), which now boil down to

$$m_1 = 0 \ , \quad m_2 = \sqrt{2} m \ , \quad (24)$$

as they should.

Problem 7 (10 points)

Let us assume that in the Standard Model there are two Higgs doublets H^1 and H^2 with the same hypercharge as the conventional Higgs. Let the fields take the following vacuum expectation values

$$H_0^1 = \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \ , \quad H_0^2 = \begin{pmatrix} v_2 \\ 0 \end{pmatrix} \ .$$

a) (4pts)Write down the unbroken generators, if there are any.

Answer : Let us first study the action of the generators on H_0^1 :

$$
\begin{aligned}
T_1 H_0^1 &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ v_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} v_1 \\ 0 \end{pmatrix} \ , \\
T_2 H_0^1 &= \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ v_1 \end{pmatrix} = -\frac{i}{2}\begin{pmatrix} v_1 \\ 0 \end{pmatrix} \ , \\
T_3 H_0^1 &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ v_1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ v_1 \end{pmatrix} \ , \\
Y H_0^1 &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ v_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ v_1 \end{pmatrix} \ .
\end{aligned} \tag{25}
$$

We immediately notice that

$$
G H_0^1 \equiv (T_3 + Y) H_0^1 = 0 \ , \tag{26}
$$

meaning that the vev of the first doublet breaks SU(2)×U(1) down to U(1). When we act with G on the second doublet, we get

$$
G H_0^2 = \begin{pmatrix} v_2 \\ 0 \end{pmatrix} \neq 0 \ , \tag{27}
$$

so the residual $U(1)$ symmetry is also broken.

Note that no other generator G' can be found, which would leave both vev's invariant.

b) (3pts)What is the unbroken group ?
Answer : There is no unbroken group.

c) (3pts)How many gauge bosons acquire mass and how many remain massless ?
Answer : Since the symmetry is completely broken, there will be 4 massive gauge bosons.