Ludwig-Maximilians-Universität München

QCD and Standard Model

Exam Solution

Prof. Dr. Georgi Dvali

Assistants : Dr. Georgios Karananas,
Emmanouil Koutsangelas, Max Warkentin

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Guidelines :

- The exam consists of 5 problems.
- The duration of the exam is 2 hours.
- Please write your name or matriculation number on every sheet that you hand in.
- Your answers should be comprehensible and readable.

GOOD LUCK !

Exercise 1	12 P
Exercise 2	8 P
Exercise 3	20 P
Exercise 4	28 P
Exercise 5	32 P

Total	100 P

Problem 1 (12 points)

Let the Standard Model Higgs doublet take the following vacuum expectation value

$$H_0 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \ .$$

a) Write down the unbroken generators, if there are any. (4P)

Answer : Parametrizing the Higgs field as

$$H_0 = \begin{pmatrix} h_1 + i\, h_2 \\ h_3 + i\, h_4 \end{pmatrix}_0 \ , \tag{1}$$

with $(h_1, \ldots h_4)_0$ real, we see that the vacuum manifold is described by

$$H_0^\dagger H_0 = \sum h_{i,0}^2 = v_1^2 + v_2^2 = \text{const.} \equiv v^2 \ . \tag{2}$$

This is nothing else than S^3, so the situation is similar to the normal case. However, the unbroken generator looks different in this parametrization. The condition for a generator to be unbroken is 1P

$$T H_0 = 0 \ . \tag{3}$$

This is solved by 2P

$$T \propto \begin{pmatrix} -\frac{v_2}{v_1} & 1 \\ 1 & -\frac{v_1}{v_2} \end{pmatrix} \ . \tag{4}$$

Notice that the above is Hermitian, i.e. $T^\dagger = T$, as it has to be. 1P

b) What is the unbroken group ? (4P)

Answer : The unbroken group is $U(1)$. 4P

c) How many gauge bosons acquire mass and how many remain massless ? (4P)

Answer : Since only one generator is unbroken and three broken, there will be one massless and three massive gauge bosons. 4 P

Problem 2 (8 points)

Consider the limit in which all the gauge and Yukawa couplings in the Standard Model are zero. What would be the symmetry of the Higgs sector in this case ? (8P)

Answer : In this case the Lagrangian of H would be

$$\mathcal{L} = \frac{1}{2}(\partial_\mu H)^\dagger \partial^\mu H - \frac{\lambda}{4}\left(H^\dagger H - v^2\right)^2 \, . \tag{5}$$

Writing

$$H = \begin{pmatrix} h_1 + i\, h_2 \\ h_3 + i\, h_4 \end{pmatrix} \, , \tag{6}$$

with $h_1, \ldots h_4$ reals, we find

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{4} \partial_\mu h_i \partial^\mu h_i - \frac{\lambda}{4}\left(\sum_{i=1}^{4} h_i^2 - v^2\right)^2 \, . \tag{7}$$

This Lagrangian is symmetric under a global SO(4) = SU(2) × SU(2) symmetry (Custodial), under which the Higgs transforms as $h_i \to h'_i = R_{ij} h_j$ with rotation matrices R. 2P for only $U(1)$, 4P for only $SU(2)$, 8P for full symmetry.

Problem 3 (20 points)

a) Demonstrate that the hypercharge is free from gauge anomalies. Consider $[\mathrm{U}(1)]^3$, as well as the mixed anomalies including hypercharge with SU(2) and SU(3). (12P) [3 P each, 1 P for formula, 2 P for calculation. In the case with one SU(N), 3 P for the argument]

Answer :

— $[U(1)]^3$: In the diagrammatic language the pure hypercharge anomaly comes from the triangle diagram with a U(1) gauge boson at each leg. This can be split as a sum of two triangle diagrams, one with the left handed fermions in the loop and one with the right handed fermions. Note that the right handed diagram comes with an overall minus due to the different sign in the projector P_R. Since the coupling is universal the diagrams do not differ up to the vertex factors, i.e. the whole triangle diagram is proportional to

$$\sum_L Y_L^3 - \sum_R Y_R^3 \, . \tag{8}$$

So for the theory to be free from the anomaly, this factor needs to vanish. The hypercharges of the fermions in the Standard Model follow from $Q = I_3 + Y/2$, i.e.

	Y_L		Y_R
u_L	$\frac{1}{3}$	u_R	$\frac{4}{3}$
d_L	$\frac{1}{3}$	d_R	$-\frac{2}{3}$
ν_L	-1	e_R	-2
e	-1		

Taking into account that the quarks appear with three different colors, we can easily check that the hypercharge anomaly vanishes :

$$\sum_L Y_L^3 = 3\left(\frac{1}{3}\right)^3 + 3\left(\frac{1}{3}\right)^3 + (-1)^3 + (-1)^3 = -\frac{16}{9} \tag{9}$$

$$\sum_R Y_R^3 = 3\left(\frac{4}{3}\right)^3 + 3\left(-\frac{2}{3}\right)^3 + (-2)^3 = -\frac{16}{9} \tag{10}$$

— The anomalies with $SU(N) \times [U(1)]^2$ vanish, because the generators of $SU(N)$ are traceless.

— $U(1)[SU(2)]^2$: Anomaly proportional to

$$\sum_{\text{l.h. quarks}} Y_L - \sum_{\text{r.h. quarks}} Y_R = 3\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right) - 3\left(\frac{4}{3} - \frac{2}{3}\right) = 0\,.$$

— $U(1) \times [SU(3)]^2$: Anomaly proportional to

$$\sum_L Y_L = 3\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right) - 1 - 1 = 0\,. \tag{11}$$

b) Consider a (gauge) U(1) theory with a massless gauge boson and 3 Dirac fermions with masses $m_1 = 4m_2 = \frac{5}{3}m_3 \neq 0$. What is the $[\mathrm{U}(1)]^3$ gauge anomaly in this case ? (8P)

<u>Answer</u> : Since the theory does not have a chiral symmetry, it cannot have an associated anomaly per definition. 8P

Problem 4 (28 points)

Assume that the mass matrices for the up- and down- type quarks have the following forms (in the basis of weak interaction eigenstates)

$$M^{(u)} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \quad \text{and} \quad M^{(d)} = m\begin{pmatrix} 1+a^2 & ab & 0 \\ ab & 1+b^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. Here $m_i, [i = u, c, t]$ the mass of the respective quark flavor, m a parameter with dimensions of mass, and a, b real.

a) Find the CKM matrix. How many independent parameters does it have? Parametrize them in terms of a and b. (18P) .

Answer : The CKM matrix appears in the W boson interaction $W^{\mu+}\bar{u}_L\gamma_\mu d_L$ after going from the weak interaction basis to the mass eigenstate basis. This happens since this interaction is flavor violating and the different flavors are rotated independently from each other. Hence, the CKM matrix is $V_{CKM} = (L^u)^\dagger L^d$, where L^u, L^d are the transformation matrices of u_L, d_L respectively.

Since $M^{(u)}$ is already diagonal in the weak eigenbasis, the CKM matrix is just L^d 2P. In order to find L^d we diagonalize $M^{(d)}$. Since $M^{(d)}$ is hermitian, it can be diagonalized with just one unitary matrix, namely V_{CKM} 2P. First, we find the eigenvalues of the matrix by considering

$$\det(M^{(d)} - \lambda \mathcal{I}_{3\times3}) = (m-\lambda)\left[(1+a^2+b^2)m - \lambda\right](m-\lambda) = 0 \ ,$$

meaning that

$$\lambda_1 = m \ , \quad \lambda_2 = m(1+a^2+b^2) \ , \quad \lambda_3 = m \ .$$

Next, we find the normalized eigenvector corresponding to the λ_i's

$$\vec{e}_1 = (0,0,1)^T \ , \quad \vec{e}_2 = \frac{1}{\sqrt{a^2+b^2}}(a,b,0)^T \ , \quad \vec{e}_3 = \frac{1}{\sqrt{a^2+b^2}}(-b,a,0)^T \ .$$

The CKM matrix V_{CKM} is such that $M^{(d)} = V_{CKM} M^{(d)}_{diag} V^\dagger_{CKM}$, with

$$M^{(d)}_{diag} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \ .$$

It is well known that such a matrix comprises the normalized eigenvectors of $M^{(d)}$, i.e. 7P

$$V_{CKM} = (\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3) = \begin{pmatrix} 0 & \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ 0 & \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \\ 1 & 0 & 0 \end{pmatrix} \ .$$

Comparing the above with the following rotation matrix

$$\begin{pmatrix} 0 & \cos\theta_{\text{mix}} & -\sin\theta_{\text{mix}} \\ 0 & \sin\theta_{\text{mix}} & \cos\theta_{\text{mix}} \\ 1 & 0 & 0 \end{pmatrix} \ ,$$

we find that the mixing angle is $\theta_{\text{mix}} = \arctan(b/a)$ 7P.

b) Will there be a physical CP-violating phase? Explain. (10 P)
Answer : There is no physical CP-violating phase since the CKM matrix given here is real. 10P

Problem 5 (32 points)

Let us now restrict ourselves to two generations of quarks. Take the mass matrix of the up-type quarks to be diagonal, and the one for the down-type quarks to be the following

$$M^{(d)} = m \begin{pmatrix} 0 & a \\ a & 2b \end{pmatrix} ,$$

with m a parameter with dimensions of mass and a, b real with $a \ll b$.

a) Find the 2×2 analog of the CKM matrix in terms of a and b. (8 P)
Answer : The eigenvalues of the mass matrix are

$$\lambda_1 = m(b+\sqrt{a^2+b^2}) \approx 2mb\,, \quad \lambda_2 = m(b-\sqrt{a^2+b^2}) \approx -\frac{a^2 m}{2b}\,. \tag{12}$$

The CKM matrix V_{CKM} is such that $M^{(d)} = V_{CKM} M^{(d)}_{diag} V^{\dagger}_{CKM}$, with

$$M^{(d)}_{diag} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} . \tag{13}$$

It is easy to see that

$$V_{CKM} = \frac{1}{\sqrt{a^2+4b^2}} \begin{pmatrix} 2b & a \\ -a & 2b \end{pmatrix} = \begin{pmatrix} \cos\theta_{\rm mix} & \sin\theta_{\rm mix} \\ -\sin\theta_{\rm mix} & \cos\theta_{\rm mix} \end{pmatrix} , \tag{14}$$

with $\theta_{\rm mix} = \arctan(a/2b)$ 4P+4P.

b) Take $m_s/m_d \approx 20$ and compare the value of the mixing angle with its experimentally measured value $\theta_{\rm mix} \approx 13^o$. (4P)
Answer : We have 3P

$$m_s = \lambda_1 \approx 2mb\,, \quad m_d = \lambda_2 \approx -\frac{a^2 m}{2b}\,,$$

meaning that $\theta_{\rm mix} \approx \arctan\sqrt{\frac{m_d}{m_s}} \approx 12.5^o$, very close to its experimentally-measured value 1P .

c) Compute the following tree-level ratios of the W- and Z- boson decay rates to quarks as a function of $\theta_{\rm mix}$

$$\frac{\Gamma(W \to ud)}{\Gamma(W \to us)}\ , \quad \frac{\Gamma(Z \to u_L u_L)}{\Gamma(Z \to d_L d_L)}\ , \quad \frac{\Gamma(Z \to u_R u_R)}{\Gamma(Z \to d_L d_L)}\ .$$

Assumptions : Take the W- and Z- bosons at rest. Assume that the quark masses are negligible compared to their energies. (20 P)

Answer :

—[W decay] (8P total) : 3P for the computation, 5P for the result

The matrix element for the $W \to ud$ decay reads

$$\begin{aligned}
\mathcal{M}_{ud} &= \frac{g}{\sqrt{2}} V_{ud} \epsilon^*_\mu \bar{u}(p) \gamma^\mu L v(q) \\
\mathcal{M}^\dagger_{ud} &= \frac{g}{\sqrt{2}} V^*_{ud} \epsilon_\mu \bar{v}(q) R \gamma^\mu u(p)
\end{aligned}$$

with spinors u and v. Putting these together we obtain

$$\begin{aligned}
|\mathcal{M}_{ud}|^2 = \sum_{\text{spins}} \mathcal{M}^\dagger \mathcal{M} &= \frac{g^2}{2} |V_{ud}|^2 \epsilon_\mu \epsilon^*_\nu q_\alpha p_\beta \mathrm{Tr}(\gamma^\mu L \gamma^\alpha \gamma^\nu L \gamma^\beta) \\
&= g^2 |V_{ud}|^2 [(q \cdot \epsilon)(p \cdot \epsilon^*) - (\epsilon^* \cdot \epsilon)(p \cdot q) + (q \cdot \epsilon^*)(p \cdot \epsilon) \\
&- i \epsilon_{\alpha\mu\beta\nu} q^\alpha \epsilon^\mu p^\beta \epsilon^{*\nu}]
\end{aligned}$$

In principle we need to calculate all polarizations and average for an unpolarized vector boson, but the unpolarized decay rate is equal to any other polarization rate since we can choose our coordinate system freely. In the following we choose it such that

$$\epsilon^\mu(0) = (0,0,0,1)^{\mathrm{T}}\ , \tag{15}$$
$$\tag{16}$$

and

$$p^\mu = m_W/2\ (1, \sin\theta, 0, \cos\theta)^{\mathrm{T}} \tag{17}$$
$$q^\mu = m_W/2\ (1, -\sin\theta, 0, -\cos\theta)^{\mathrm{T}} \tag{18}$$
$$\tag{19}$$

We observe that the Levi-Civita-term vanishes due to the symmetry of $\epsilon^\mu \epsilon^{*\nu}$ under $\mu \leftrightarrow \nu$. Further, $\epsilon^* \cdot \epsilon = -1$, so that we get

$$|\mathcal{M}_{ud}(0)|^2 = \frac{g^2 m_W^2}{2} |V_{ud}|^2 \sin^2\theta \ , \tag{20}$$

where θ is the angle between $\vec{p}$ and the z-axis. The partial decay and the total decay rate are then

$$\begin{aligned} \frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega}(W \to ud) &= \frac{g^2 m_W}{64\pi^2} |V_{ud}|^2 \frac{1}{2} \sin^2\theta \\ \Gamma(W \to ud) &= \frac{g^2 m_W}{48\pi} |V_{ud}|^2 \ . \end{aligned}$$

For the $W \to us$ decay, a completely analogous computation reveals that

$$\Gamma(W \to us) = \frac{g^2 m_W}{48\pi} |V_{us}|^2 \ .$$

Using $V_{ud} = V_{11} = \cos\theta_{\text{mix}}$ and $V_{us} = V_{12} = \sin\theta_{\text{mix}}$, we find the ratio

$$\frac{\Gamma(W \to ud)}{\Gamma(W \to us)} = \cot^2\theta_{\text{mix}} \ .$$

—[Z decay] (6P + 6P) : 2P + 2P for the computations, 4P+4P for the results
The interaction term responsible for the $Z \to u\ u$ decay is

$$\begin{aligned} \mathcal{L} \supset g Z^\mu \Bigg[\left(\cos\theta_W I^3 - \sin\theta_W \tan\theta_W \frac{Y_Q}{2} \right) \bar{u}_L \gamma_\mu u_L \\ - \left(\sin\theta_W \tan\theta_W \frac{Y_u}{2} \right) \bar{u}_R \gamma_\mu u_R \Bigg] \\ \equiv g Z^\mu [c_L\, \bar{u}_L \gamma_\mu u_L + c_R\, \bar{u}_R \gamma_\mu u_R] \end{aligned}$$

Notice that the Z- boson couplings to left and right handed quarks are not equal.
With the above interaction terms we can perform practically the same calculation as in the W-boson decay. There are only two differences : The first difference is that the interaction term has no factor of $1/\sqrt{2}$, The second difference, which is the most important, is that the CKM

matrix does not appear at tree level. The total decay rates are found to be

$$\Gamma(Z \to u_L u_L) = \frac{g^2 m_Z}{24\pi} c_L^2 \ , \quad \Gamma(Z \to u_R u_R) = \frac{g^2 m_Z}{24\pi} c_R^2 \ .$$

Similarly, for the down quarks,

$$\Gamma(Z \to d_L d_L) = \frac{g^2 m_Z}{24\pi} \tilde{c}_L^2 \ , \quad \Gamma(Z \to d_R d_R) = \frac{g^2 m_Z}{24\pi} \tilde{c}_R^2 \ .$$

We notice that there is no dependence on the mixing angle $\theta_{\rm mix}$, something of course expected. Putting the numbers for c_i's and $\tilde{c}_i$'s $[i = L, R]$, we find that the ratios are

$$\frac{\Gamma(Z \to u_L u_L)}{\Gamma(Z \to d_L d_L)} = \left(\frac{c_L}{\tilde{c}_L}\right)^2 \approx 0.66 \ , \quad \frac{\Gamma(Z \to u_R u_R)}{\Gamma(Z \to d_L d_L)} = \left(\frac{c_R}{\tilde{c}_L}\right)^2 \approx 0.15 \ .$$