Ludwig-Maximilians-Universität München

SOLUTIONS TO

QCD and Standard Model

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Guidelines :

- The exam consists of 7 problems.
- The duration of the exam is 24 hours.
- Please write your name or matriculation number on every sheet that you hand in.
- Your answers should be comprehensible and readable.

GOOD LUCK !

Problem 1	14 P
Problem 2	12 P
Problem 3	12 P
Problem 4	21 P
Problem 5	17 P
Problem 6	8 P
Problem 7	16 P

Total	100 P

Problem 1 (14 points)

Consider a theory invariant under a global SU(2) symmetry with two scalar fields, Σ and Φ, each transforming in the adjoint representation of the group, i.e.

$$\Sigma \to \Sigma' = U\Sigma U^\dagger \ ,$$
$$\Phi \to \Phi' = U\Phi U^\dagger \ ,$$

where $U \in SU(2)$ and $\dagger$ stands for Hermitian conjugation.

a) Assume that the scalar fields do not interact with each other. Then the most general potential for each scalar field could involve the invariants

   i) $\mathrm{tr}Z$,

   ii) $\mathrm{tr}Z^2$,

   iii) $\mathrm{tr}Z^3$,

   iv) $\mathrm{tr}Z^4$,

   v) $(\mathrm{tr}Z^2)^2$,

   vi) $\det(Z)$,

   vii) $(\det(Z))^2$,

   with $Z = \Sigma, \Phi$. Which of the above invariants are identically zero? How many of non-zero invariants are independent? (*Hint : Bring the fields into a simple form by a group transformation.*)
   Use this, to write down the Lagrangian for Σ and Φ.

   Sol. [3P] As Z transform under the adjoint representation, it can be written as

$$Z = \lambda^a T^a \equiv \frac{1}{2}\begin{pmatrix} \lambda^3 & \lambda^1 - i\lambda^2 \\ \lambda^1 + i\lambda^2 & -\lambda^3 \end{pmatrix}, \tag{1}$$

   where $a = 1, 2, 3$, and T^a are the generators of $SU(2)$. Using the identities :

$$\mathrm{tr}T^i = 0,$$
$$\mathrm{tr}T^iT^j = \frac{1}{2}\delta^{ij},$$
$$T^jT^k = \frac{1}{4}\left(\delta^{jk}I + 2i\varepsilon^{jk\ell}T^\ell\right),$$

   we have

   i)

$$\mathrm{tr}Z = \lambda^a\,\mathrm{tr}T^a = 0,$$

ii)

$$\mathrm{tr}Z^2 = \mathrm{tr}(\lambda^a T^a \lambda^b T^b) = \lambda^a \lambda^b \,\mathrm{tr}T^a T^b = \frac{1}{2}\lambda^a \lambda^a,$$

iii)

$$\begin{aligned}\mathrm{tr}Z^3 &= \mathrm{tr}(\lambda^a T^a \lambda^b T^b \lambda^c T^c) = \lambda^a \lambda^b \lambda^c \,\mathrm{tr}(T^a T^b T^c)\\ &= \frac{1}{4}\lambda^a \lambda^b \lambda^c \,\mathrm{tr}(\left(\delta^{ab} I + 2i\varepsilon^{ab\ell}\, T^\ell\right) T^c)\\ &= \frac{1}{4}\lambda^a \lambda^a \lambda^c \,\mathrm{tr}(T^c) + i\frac{1}{2}\lambda^a \lambda^b \lambda^c \varepsilon^{ab\ell} \,\mathrm{tr}T^\ell T^c\\ &= i\frac{1}{4}\lambda^a \lambda^b \lambda^c \varepsilon^{abc} = 0,\end{aligned}$$

iv)

$$\begin{aligned}\mathrm{tr}Z^4 &= \mathrm{tr}(\lambda^a T^a \lambda^b T^b \lambda^c T^c \lambda^d T^d)\\ &= \lambda^a \lambda^b \lambda^c \lambda^c \,\mathrm{tr}(T^a T^b T^c T^d)\\ &= \frac{1}{16}\lambda^a \lambda^b \lambda^c \lambda^d \,\mathrm{tr}(\left(\delta^{ab} I + 2i\varepsilon^{ab\ell}\, T^\ell\right) \left(\delta^{cd} I + 2i\varepsilon^{cdm}\, T^m\right))\\ &= \frac{1}{16}\lambda^a \lambda^b \lambda^c \lambda^d \left(2\delta^{ab}\delta^{cd} - 4\varepsilon^{ab\ell}\varepsilon^{cdm} \,\mathrm{tr}T^\ell T^m\right)\\ &= \frac{1}{16}\lambda^a \lambda^b \lambda^c \lambda^d \left(2\delta^{ab}\delta^{cd} - 2\varepsilon^{ab\ell}\varepsilon^{cd\ell}\right)\\ &= \frac{1}{16}\lambda^a \lambda^b \lambda^c \lambda^d \left(2\delta^{ab}\delta^{cd} - 2(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc})\right)\\ &= \frac{1}{8}\lambda^a \lambda^a \lambda^c \lambda^c\\ &= \frac{1}{8}(\lambda^a \lambda^a)^2,\end{aligned}$$

v)

$$\left(\mathrm{tr}Z^2\right)^2 = \left(\frac{1}{2}\lambda^a \lambda^a\right)^2 = \frac{1}{4}(\lambda^a \lambda^a)^2,$$

vi)

$$\det(Z) = -\frac{1}{4}\left(\lambda^a \lambda^a\right),$$

vii)

$$(\det(Z))^2 = \left(-\frac{1}{4}\left(\lambda^a \lambda^a\right)\right)^2 = \frac{1}{16}\left(\lambda^a \lambda^a\right)^2,$$

We conclude that at most two non-zero invariants are linearly independent, e.g. $\mathrm{tr}Z^2$ and $(\mathrm{tr}Z^2)^2$.
The Lagrangian for Σ and Φ reads :

$$\mathcal{L} = \mathrm{tr}(\partial_\mu \Sigma \partial^\mu \Sigma) + \mathrm{tr}(\partial_\mu \Phi\, \partial^\mu \Phi) - V(\Sigma, \Phi),$$

with

$$V(\Sigma, \Phi) = m_\Sigma^2 \,\mathrm{tr}\Sigma^2 + \lambda_\Sigma \left(\mathrm{tr}\Sigma^2\right)^2 + m_\Phi^2 \,\mathrm{tr}\Phi^2 + \lambda_\Phi \left(\mathrm{tr}\Phi^2\right)^2 . \qquad (2)$$

b) Arrange the potential such that the vacuum expectation value (vev) of each scalar field is non-zero. Find the vev by explicitly minimizing the potential.

Sol. [3P] The potential 2 can be rewritten as

$$V(\Sigma, \Phi) = \lambda_1 \left(\mathrm{tr}\Sigma^2 - \frac{v_1^2}{2}\right)^2 + \lambda_2 \left(\mathrm{tr}\Phi^2 - \frac{v_2^2}{2}\right)^2 .$$

where $\lambda_1 = \lambda_\Sigma$, $\lambda_2 = \lambda_\Phi$, $v_1 = \frac{m_\Sigma}{\sqrt{\lambda_\Sigma}}$, and $v_2 = \frac{m_\Phi}{\sqrt{\lambda_\Phi}}$. Rewriting $\Phi = \phi^a T^a$ and $\Sigma = \varphi^a T^a$,

$$V(\varphi^a, \phi^a) = \frac{\lambda_1}{4} \left(\varphi^a \varphi^a - v_1^2\right)^2 + \frac{\lambda_2}{4} \left(\phi^a \phi^a - v_2^2\right)^2 .$$

Minimizing the potential :

$$\frac{\partial V}{\partial \varphi^a} = \lambda_1 (\varphi^c \varphi^c - v_1^2) \varphi^a = 0,$$

implies

$$\varphi^c \varphi^c = v_1^2,$$

at the minimum. Similarly for Φ, we get

$$\phi^c \phi^c = v_2^2.$$

c) How many generators are broken? Does the answer depend on the "direction" of the vacuum?

Sol. [1P] If $\varphi^a \parallel \phi^a$ there are two broken generators and the $SU(2)$ symmetry is spontaneously broken to $U(1)$. If the $\varphi^a \nparallel \phi^a$ there are three broken generators and the $SU(2)$ symmetry is completely broken.

d) Assume now that the global SU(2) is made local. Construct the appropriate covariant derivatives for Σ and Φ.

Sol. [3P] The derivative of Z transforms as

$$\begin{aligned}\partial_\mu Z \to \partial_\mu(Z') &= \partial_\mu(UZU^\dagger) \ , \\ &= (\partial_\mu U)ZU^\dagger + U(\partial_\mu Z)U^\dagger + UZ(\partial_\mu U^\dagger) \ ,\end{aligned}$$

for an infinitesimal transformation $U = I + i\alpha^a T^a$

$$\begin{aligned}\partial_\mu U &= i\partial_\mu \alpha^a T^a, \\ \partial_\mu U^\dagger &= -i\partial_\mu \alpha^a T^a,\end{aligned}$$

$$\partial_\mu Z \to (i\partial_\mu \alpha^a T^a)Z + (I + i\alpha^a T^a)(\partial_\mu Z)(I - i\alpha^a T^a) + Z(-i\partial_\mu \alpha^a T^a).$$

Thus

$$\delta\partial_\mu Z = i(\partial_\mu \alpha^a)[T^a, Z] + i\alpha^a[T^a, \partial_\mu Z].$$

From here we construct the covariant derivative as follows

$$D_\mu Z \equiv \partial_\mu Z + \mathrm{ad}(A_\mu)Z = \partial_\mu Z + [A_\mu, Z], \tag{3}$$

as we require that the covariant derivative to transforms as

$$D_\mu Z \to U(D_\mu Z)U^\dagger.$$

To see this is the case, we recall that the gauge field A_μ transforms as

$$A_\mu \to A'_\mu = UA_\mu U^\dagger + U\partial_\mu U^\dagger \tag{4}$$

and for an infinitesimal transformation

$$\delta A_\mu = i\alpha^a[T^a, A_\mu] - i\partial_\mu \alpha^a T^a$$

Then, we can check that the covariant derivative 3 transforms to

$$\begin{aligned}(D_\mu Z)' =& (\partial_\mu U)ZU^\dagger + U(\partial_\mu Z)U^\dagger + UZ(\partial_\mu U^\dagger) \\ &+ [UA_\mu U^\dagger + U\partial_\mu U^\dagger, UZU^\dagger] \\ =& U(\partial_\mu Z)U^\dagger + [UA_\mu U^\dagger, UZU^\dagger] \\ =& U(\partial_\mu Z + [A_\mu, Z])U^\dagger, \\ =& U(D_\mu Z)U^\dagger\end{aligned}$$

Note that the gauge field A_μ takes values in the $SU(2)$ algebra, and can be expressed in terms of three real fields A^a_μ

$$A_\mu \equiv -igA^a_\mu T^a,$$

and the covariant derivative can be rewritten as

$$(D_\mu Z)^a = \partial_\mu \lambda^a + g\varepsilon^{abc} A^b_\mu \lambda^c,$$

e) Compute the masses of the gauge bosons on top of the vacua you found in b). Is it possible to find that all gauge bosons have different masses ? Explain.

Sol. [4P] The lagrangian for Σ and Φ is now

$$\mathcal{L} = \operatorname{tr}(D_\mu \Sigma D^\mu \Sigma) + \operatorname{tr}(D_\mu \Phi\, D^\mu \Phi) - V(\Sigma, \Phi).$$

The mass terms for the vector bosons comes from the terms

$$\mathcal{L}_M = \frac{g^2}{2}(\varepsilon^{abc} A^b_\mu \lambda^c)(\varepsilon^{ab'c'} A^{b'}_\mu \lambda^{c'}) \tag{5}$$

in the Lagrangian. If $\varphi^a \parallel \phi^a$ there are two broken generators and the $SU(2)$ symmetry is spontaneously broken to $U(1)$. In this case we can choose a unitary gauge in which

$$\begin{aligned} \varphi^a &= v_1 \delta^a 3 \\ \phi^a &= v_2 \delta^a 3. \end{aligned}$$

Then the mass term 5 reads as :

$$\begin{aligned} \mathcal{L}_M &= \frac{g^2}{2}(v_1^2 + v_2^2)(\varepsilon^{abc} A^b_\mu \delta^{c3})(\varepsilon^{ab'c'} A^{b'}_\mu \delta^{c'3}) \\ &= \frac{g^2}{2}(v_1^2 + v_2^2)(A^b_\mu A^b_\mu - A^3_\mu A^3_\mu). \end{aligned}$$

In this case A_3 is massless, and A_1 and A_2 have masses $m = g\sqrt{v_1^2 + v_2^2}$. If $\varphi^a \nparallel \phi^a$ there are three broken generators and the $SU(2)$ symmetry is completely broken. We can choose a unitary gauge in which $\varphi^a = v_1 \delta^a 3$ and $\phi^a = v_2 n^a$, where n^a is a unitary vector parametrized by the angle θ between the directions of φ^a and ϕ^a as :

$$
\begin{aligned}
n^1 &= \sin\theta, \\
n^2 &= 0, \\
n^3 &= \cos\theta.
\end{aligned}
$$

Then the mass term 5 reads as :

$$
\begin{aligned}
\mathcal{L}_M &= \frac{g^2}{2} v_1^2 (\varepsilon^{abc} A_\mu^b \delta^{c3})(\varepsilon^{ab'c'} A_\mu^{b'} \delta^{c'3}) + \frac{g^2}{2} v_2^2 (\varepsilon^{abc} A_\mu^b n^c)(\varepsilon^{ab'c'} A_\mu^{b'} n^{c'}) \\
&= \frac{g^2}{2} v_1^2 (A_\mu^b A_\mu^b - A_\mu^3 A_\mu^3) + \frac{g^2}{2} v_2^2 (A_\mu^b A_\mu^b - (A_\mu^1 \sin\theta + A_\mu^3 \cos\theta)^2) \\
&= \frac{g^2}{2} \Big((v_1^2 + v_2^2 \cos^2\theta) {A_\mu^1}^2 + (v_1^2 + v_2^2) {A_\mu^2}^2 + v_2^2 \sin^2\theta {A_\mu^3}^2 \\
&\quad - 2 v_2^2 \sin\theta \cos\theta A_\mu^1 A_\mu^3 \Big).
\end{aligned}
$$

The mass of A_μ^2 is again $m = g\sqrt{(v_1^2 + v_2^2)}$. However, we need to diagonalize the mass terms for A_1 and A_3,i.e. :

$$
\frac{g^2}{2} (A_\mu^1, A_\mu^3) \begin{pmatrix} v_1^2 + v_2^2 \cos^2\theta & -v_2^2 \sin\theta\cos\theta \\ -v_2^2 \sin\theta \cos\theta & v_2^2 \sin^2\theta \end{pmatrix} \begin{pmatrix} A_\mu^1 \\ A_\mu^3 \end{pmatrix},
$$

After computing the eigenvalues, we get that the masses square are

$$
m_\pm^2 = \frac{g^2}{2} \left(v_1^2 + v_2^2 \pm \sqrt{2 v_2^2 v_1^2 \cos(2\theta) + v_1^4 + v_2^4} \right).
$$

In the case $\theta = \pi/2$, the three gauge fields have different masses :

$$
\begin{aligned}
m_1 &= g v_1 \\
m_2 &= g\sqrt{(v_1^2 + v_2^2)} \\
m_3 &= g v_2.
\end{aligned}
$$

Problem 2 (12 points)

Consider a theory invariant under a global SU(N), $N > 4$ symmetry. Take X to be a symmetric $N \times N$ matrix scalar field that under an SU(N) transformation behaves as

$$X \to X' = UXU^T \ ,$$

where $U \in SU(N)$ and T stands for transpose.

a) Write down the most general, renormalizable, SU(N)- and Lorentz-invariant Lagrangian for X in four spacetime dimensions.

Sol. [4P] Lets begin by constructing $SU(N)$-invariants.

$$X \to X' = UXU^T \ ,$$

implies

$$X^* \to X'^* = U^* X^* U^\dagger \ ,$$

We notice also that $X = X^T$ and $X^* = X^\dagger$. From here, it is easy to show that $\mathrm{tr}[XX^\dagger]$ is invariant :

$$\mathrm{tr}[XX^\dagger] \to \mathrm{tr}[X'X'^\dagger] = \mathrm{tr}[UXU^TU^*X^\dagger U^\dagger] = \mathrm{tr}[XX^\dagger] \ .$$

Similarly $(\mathrm{tr}[XX^\dagger])^2$ and $\mathrm{tr}[XX^\dagger XX^\dagger]$ are invariant. No cubic term in X is invariant since an even power of X is required. Finally no det is considered as $N > 4$. The most general Lagrangian is then :

$$\mathcal{L} = \mathrm{tr}[(\partial_\mu X)(\partial^\mu X)^\dagger] - m^2 \ \mathrm{tr}[XX^\dagger] - \lambda \ \mathrm{tr}[XX^\dagger XX^\dagger] - \gamma \ (\mathrm{tr}[XX^\dagger])^2$$

b) Assume now that we gauge the SU(N) symmetry. Construct the appropriate covariant derivative for X. If you just "guess" it without a derivation, you have to show that it indeed transforms as a covariant derivative.

Sol. [8P] The derivative of X transforms as

$$\begin{aligned}\partial_\mu X \to \partial_\mu(X') &= \partial_\mu(UXU^T) \ , \\ &= (\partial_\mu U)XU^T + U(\partial_\mu X)U^T + UX(\partial_\mu U^T) \ ,\end{aligned}$$

for an infinitesimal transformation $U = I + i\alpha^a T^a$, where T^a are the generators of the $SU(N)$ Lie algebra and $a = 1, \ldots, N^2 - 1$:

$$\begin{aligned}\partial_\mu U &= i\partial_\mu \alpha^a T^a, \\ \partial_\mu U^T &= i\partial_\mu \alpha^a T^{aT} = i\partial_\mu \alpha^a T^{a*},\end{aligned}$$

$$\partial_\mu X \rightarrow (i\partial_\mu \alpha^a T^a)X + (I + i\alpha^a T^a)(\partial_\mu X)(I + i\alpha^a T^{aT}) + X(i\partial_\mu \alpha^a T^{aT}).$$

Thus

$$\begin{aligned}
\delta\partial_\mu X &= i\partial_\mu \alpha^a T^a X + i\partial_\mu \alpha^a X T^{aT} + i\alpha^a T^a \partial_\mu X + i\alpha^a \partial_\mu X T^{aT} \\
&= 2i\partial_\mu \alpha^a (T^a X)_S + 2i\alpha^a (T^a \partial_\mu X)_S,
\end{aligned}$$

where $(Y)_S = \frac{Y+Y^T}{2}$. From here we construct the covariant derivative as follows

$$\begin{aligned}
D_\mu X &= \partial_\mu X + \mathrm{sym}(A_\mu)X \\
&= \partial_\mu X + 2(A_\mu X)_S \\
&= \partial_\mu X + A_\mu X + X A_\mu^T,
\end{aligned}$$

Then, recalling equation 4 we can check that the covariant derivative transforms to

$$\begin{aligned}
(D_\mu X)' &= (\partial_\mu U) X U^T + U(\partial_\mu X)U^T + UX(\partial_\mu U^T) \\
&\quad + 2\left((UA_\mu U^\dagger + U\partial_\mu U^\dagger)(UXU^T)\right)_S \\
&= (\partial_\mu U) X U^T + U(\partial_\mu X)U^T + UX(\partial_\mu U^T) \\
&\quad + 2\left(UA_\mu X U^T\right)_S + 2\left(U\partial_\mu U^\dagger U X U^T\right)_S \\
&= U\left(\partial_\mu X + 2\left(A_\mu X\right)_S\right)U^T + (\partial_\mu U) X U^T + UX(\partial_\mu U^T) \\
&\quad - 2\left(UU^\dagger \partial_\mu U X U^T\right)_S \\
&= U\left(\partial_\mu X + 2\left(A_\mu X\right)_S\right)U^T, \\
&= U\left(D_\mu X\right)U^T
\end{aligned}$$

Note that the gauge field A_μ takes values in the $SU(N)$ algebra, and can be expressed in terms of $N^2 - 1$ real fields A_μ^a

$$A_\mu \equiv -igA_\mu^a T^a,$$

and the covariant derivative can be rewritten as

$$\begin{aligned}
D_\mu X &= \partial_\mu X - 2igA_\mu^a \left(T^a X\right)_S \\
&= \partial_\mu X - igA_\mu^a \left(T^a X + X T^{aT}\right)
\end{aligned}$$

Problem 3 (12 points)

Focus only on the leptonic sector of the Standard Model supplemented with three right-handed neutrinos.

a) Take the Yukawa couplings to be zero. What are the global symmetries (apart from the gauged $SU(2) \times U(1)$) ?

Sol. [3P] In the absence of Yukawa couplings, the leptonic sector consists only of the kinetic terms for the fields, i.e.

$$\mathcal{L}_{\mathrm{L}} = i(L_L)^a \gamma^\mu D_\mu (L_L)_a + i(e_R)^a \gamma^\mu D_\mu (e_R)_a + i(\nu_R)^a \gamma^\mu \partial_\mu (\nu_R)_a \ ,$$

possibly supplemented by a Majorana mass for the right-handed neutrinos ; in what follows, let's assume that this term is absent. Here $a = 1, 2, 3$ stands for the generation index and D_μ is the covariant derivative for the corresponding fermionic field.

From the above we see that the Lagrangian is invariant under

$$\begin{aligned} (L_L)^a &\longrightarrow (U_{L_L})^a_b (L_L)^b \\ (e_R)^a &\longrightarrow (U_{e_R})^a_b (e_R)^b \\ (\nu_R)^a &\longrightarrow (U_{\nu_R})^a_b (\nu_R)^b \ , \end{aligned}$$

where $U_{L_L}, U_{e_R}, U_{\nu_R} \in \mathrm{U}(3)$ and independent of each other. Thus, the global flavor symmetry is

$$G_{\mathrm{L}} = U(3)^3 = \mathrm{U}(3)_{L_L} \times \mathrm{U}(3)_{e_R} \times \mathrm{U}(3)_{\nu_R} \ .$$

b) What are the symmetries once we switch on arbitrary Yukawa couplings ?

Sol. [3P] In the presence of arbitrary Yukawas, there is a global U(1). (However, and this is very important, when there is a Majorana mass term, this symmetry is explicitly broken.)

c) What is the symmetry if we assume that the Yukawa couplings are non-zero diagonal matrices ?

Sol. [3P] For diagonal Yukawas, each generation rotates separately, meaning that there will be $U(1)^3$.

(As in b), when there is a Majorana mass term, these symmetries are explicitly broken.)

d) Do your answers for parts a) and b) change if the right-handed neutrinos are removed ? Explain.

Sol. [3P] Let us now remove the right-handed neutrinos.

a) We can see the difference in the kinetic terms of the leptons that now reads

$$\mathcal{L}_{\mathrm{L}} = i(L_L)^a \gamma^\mu D_\mu (L_L)_a + i(e_R)^a \gamma^\mu D_\mu (e_R)_a \ ,$$

meaning that the symmetry is now $U(3)^2$.
b) For arbitrary Yukawas, there is now a global $U(1)^3$ symmetry, which is responsible for the usual lepton flavor number conservation (lepton number is conserved for every generation).

Problem 4 (21 points)

Consider the process

$$e^-(p_1)e^+(p_2) \to \mu^-(p_3)\mu^+(p_4)\ ,$$

and assume that all fermion masses are zero.

In the Standard Model, at tree-level, this can occur through the exchange of a photon or Z boson (you can neglect the Higgs boson exchange in the zero-fermion-mass limit). The interaction of charged leptons with photons is given by the vertex $ie\gamma_\mu$, while their interaction with Z-bosons is given by $ie/(2\sin(2\theta_W))(g_V\gamma_\mu + g_A\gamma_\mu\gamma_5)$.

a) Starting from the (covariant) kinetic terms of the charged leptons, show that $g_V = (1-4\sin^2\theta_W)$ and $g_A = 1$.
Sol. [2P] The electro-weak sector of the Standard Model includes the term

$$\mathcal{L}_{EW} \supset i\sum_j \bar{\psi}_j \not{D}\psi_j \tag{6}$$

where $\psi \in \{E_L, e_R, \dots\}$ with $E_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L$.
The covariant derivative is given by

$$D_\mu = \partial_\mu - igW_\mu^a T_L^a - ig'\frac{Y}{2}B_\mu, \tag{7}$$

with

$$B_\mu = \mathrm{c}_{\theta_W} A_\mu - \mathrm{s}_{\theta_W} Z_\mu \quad \text{and} \quad W_\mu^3 = \mathrm{s}_{\theta_W} A_\mu + \mathrm{c}_{\theta_W} Z_\mu, \tag{8}$$

where we defined $\mathrm{c}_{\theta_W} \equiv \cos(\theta_W)$ and $\mathrm{s}_{\theta_W} \equiv \sin(\theta_W)$.
Therefore,

$$\mathcal{L}_{EW} \supset \bar{E}_L^j\gamma^\mu \left(gW_\mu^3 T_L^3 + g'\frac{Y}{2}B_\mu\right) E_L^j + \bar{e}_R^j\gamma^\mu g'\frac{Y}{2}B_\mu e_R^j. \tag{9}$$

Using the above expressions for B_μ and W_μ^3 we obtain

$$\begin{aligned}
&\mathcal{L}_{EW} \supset \\
&\supset Z_\mu \left[\bar{E}_L^j\gamma^\mu \left(g\mathrm{c}_{\theta_W} T_L^3 - g'\mathrm{s}_{\theta_W}\frac{Y_L}{2}\right) E_L^j - \bar{e}_R^j\gamma^\mu g'\mathrm{s}_{\theta_W}\frac{Y_R}{2} e_R^j\right] + \\
&+ A_\mu \left[\bar{E}_L^j\gamma^\mu \left(g\mathrm{s}_{\theta_W} T_L^3 + g'\mathrm{c}_{\theta_W}\frac{Y_L}{2}\right) E_L^j + \bar{e}_R^j\gamma^\mu g'\mathrm{c}_{\theta_W}\frac{Y_R}{2} e_R^j\right].
\end{aligned} \tag{10}$$

Using that $e = g\mathrm{s}_{\theta_W} = g'\mathrm{c}_{\theta_W}$, $T_L^3(e_L) = -1/2$, $Y_L(e_L) = -1$ and $Y_R(e_R) = -2$, we obtain

$$\begin{aligned}
\mathcal{L}_{EW} \supset & \frac{Z_\mu}{\mathrm{s}_{\theta_W}\mathrm{c}_{\theta_W}}\left[\bar{e}_L^j\gamma^\mu\frac{e}{2}(\mathrm{s}_{\theta_W}^2-\mathrm{c}_{\theta_W}^2)e_L^j+\bar{e}_R^j\gamma^\mu e\mathrm{s}_{\theta_W}^2 e_R^j\right]+\\
& A_\mu\left[-\bar{e}_L^j\gamma^\mu e e_L^j+\bar{e}_R^j\gamma^\mu e e_R^j\right]=\\
=& Z_\mu\frac{2e}{\mathrm{s}_{2\theta_W}}\left[\bar{e}^j\gamma^\mu\mathrm{s}_{\theta_W}^2 e^j-\frac{1}{2}\bar{e}_L^j\gamma^\mu e_L^j\right]-A_\mu\bar{e}^j e\gamma^\mu e^j=\\
=& Z_\mu\frac{e}{2\mathrm{s}_{2\theta_W}}\left[\bar{e}^j\gamma^\mu\left(4\mathrm{s}_{\theta_W}^2-(1+\gamma_5)\right)e^j\right]-A_\mu\bar{e}^j e\gamma^\mu e^j=\\
=& -Z_\mu\frac{e}{2\mathrm{s}_{2\theta_W}}\bar{e}^j\left[\left(1-4\mathrm{s}_{\theta_W}^2\right)\gamma^\mu+\gamma^\mu\gamma_5\right]e^j-A_\mu\bar{e}^j e\gamma^\mu e^j
\end{aligned} \tag{11}$$

Therefore,

$$g_V = 1 - 4\mathrm{s}_{\theta_W}^2 \quad \text{and} \quad g_A = 1. \tag{12}$$

b)+c) Split the above vertices into vertices that describe interactions of vector bosons with fermions of definite helicities. Show that the amplitude $e^+e^- \to \mu^+\mu^-$ can be written as

$$\begin{aligned}
\mathcal{M}_{Z+\gamma} \equiv \mathcal{M}_Z + \mathcal{M}_\gamma = \frac{ie^2}{s}\,[&A_{LL}\bar{v}(p_2)\gamma^\mu\omega_L u(p_1)\bar{u}(p_3)\gamma_\mu\omega_L v(p_4)+\\
+&A_{RL}\bar{v}(p_2)\gamma^\mu\omega_R u(p_1)\bar{u}(p_3)\gamma_\mu\omega_L v(p_4)+\\
+&A_{LR}\bar{v}(p_2)\gamma^\mu\omega_L u(p_1)\bar{u}(p_3)\gamma_\mu\omega_R v(p_4)+\\
+&A_{RR}\bar{v}(p_2)\gamma^\mu\omega_R u(p_1)\bar{u}(p_3)\gamma_\mu\omega_R v(p_4)\,]
\end{aligned}$$

where $\omega_{L/R} = P_{L/R} = (1 \pm \gamma_5)/2$ are projectors on different helicity states and s is the square of the center-of-mass energy. Express the coefficients A_{ij} through the couplings $g_{A,V}$.

Calculate all the relevant helicity amplitudes in terms of spinor products.

Sol. [6P]

Define for brevity : $\bar{v}_2 \equiv \bar{v}(p_2)$, $u_1 \equiv u(p_1)$, $\bar{u}_3 \equiv \bar{u}(p_3)$ and $v_4 \equiv v(p_4)$. Then $\mathcal{M}_\gamma$ is given by

$$\begin{aligned}
\mathcal{M}_\gamma &= \bar{v}_2\,(ie\gamma^\mu)\,u_1\frac{-i\eta_{\mu\nu}}{\underbrace{(p_1+p_2)^2}_{=s}}\bar{u}_3\,(ie\gamma^\nu)\,v_4\\
\Leftrightarrow \mathcal{M}_\gamma &= \frac{ie^2}{s}\bar{v}_2\gamma^\mu u_1\bar{u}_3\gamma_\mu v_4.
\end{aligned} \tag{13}$$

Using the Feynman-t'Hooft gauge for the Z boson propagator in $\mathcal{M}_Z$ we obtain

$$\mathcal{M}_Z = \bar{v}_2 \left[\frac{ie\gamma^\mu \left(g_V + g_A\gamma_5\right)}{2\mathrm{s}_{2\theta_W}} \right] u_1 \frac{-i\eta_{\mu\nu}}{s - m_Z^2} \bar{u}_3 \left[\frac{ie\gamma^\nu \left(g_V + g_A\gamma_5\right)}{2\mathrm{s}_{2\theta_W}} \right] v_4 \tag{14}$$

Or, equivalently,

$$\begin{aligned} \mathcal{M}_Z = \frac{ie^2}{4\mathrm{s}_{2\theta_W}^2} \frac{1}{s - m_Z^2} & [\bar{v}_2\gamma^\mu\gamma_5 u_1 \bar{u}_3 \gamma_\mu \gamma_5 v_4 g_A^2 + \\ & + \bar{v}_2\gamma^\mu\gamma_5 u_1 \bar{u}_3 \gamma_\mu v_4 g_A g_V + \\ & + \bar{v}_2\gamma^\mu u_1 \bar{u}_3 \gamma_\mu \gamma_5 v_4 g_V g_A + \\ & + \bar{v}_2\gamma^\mu u_1 \bar{u}_3 \gamma_\mu v_4 g_V^2 \ ] \end{aligned} \tag{15}$$

Define for brevity

$$\mathcal{M}_{ij} \equiv \bar{v}_2\gamma^\mu \omega_i u_1 \bar{u}_3 \gamma_\mu \omega_j v_4, \tag{16}$$

with $i, j \in \{L, R\}$.
Using $\omega_L + \omega_R = 1$, we obtain

$$\mathcal{M}_\gamma = \frac{ie^2}{s} \left[\mathcal{M}_{LL} + \mathcal{M}_{LR} + \mathcal{M}_{RL} + \mathcal{M}_{RR} \right] \tag{17}$$

To obtain $\mathcal{M}_Z$, using $\gamma_5 = (\omega_L - \omega_R)$, we first calculate the following

$$\begin{aligned} & \bar{v}_2\gamma^\mu\gamma_5 u_1 \bar{u}_3 \gamma_\mu \gamma_5 v_4 g_A^2 = \\ & = \bar{v}_2\gamma^\mu \left(\omega_L - \omega_R\right) u_1 \bar{u}_3 \gamma_\mu \left(\omega_L - \omega_R\right) v_4 g_A^2 = \\ & = \left(\mathcal{M}_{LL} - \mathcal{M}_{LR} - \mathcal{M}_{RL} + \mathcal{M}_{RR} \right) g_A^2 \end{aligned} \tag{18}$$

$$\bar{v}_2\gamma^\mu\gamma_5 u_1 \bar{u}_3 \gamma_\mu v_4 g_A g_V = \left(\mathcal{M}_{LL} + \mathcal{M}_{LR} - \mathcal{M}_{RL} - \mathcal{M}_{RR} \right) g_A g_V \tag{19}$$

$$\bar{v}_2\gamma^\mu u_1 \bar{u}_3 \gamma_\mu \gamma_5 v_4 g_V g_A = \left(\mathcal{M}_{LL} - \mathcal{M}_{LR} + \mathcal{M}_{RL} - \mathcal{M}_{RR} \right) g_V g_A \tag{20}$$

$$\bar{v}_2\gamma^\mu u_1 \bar{u}_3 \gamma_\mu v_4 g_V^2 = \left(\mathcal{M}_{LL} + \mathcal{M}_{LR} + \mathcal{M}_{RL} + \mathcal{M}_{RR} \right) g_V^2 \tag{21}$$

Therefore, for $\mathcal{M}_Z$ we obtain

$$\begin{aligned} \mathcal{M}_Z = \frac{ie^2}{s} \frac{s}{4\mathrm{s}_{2\theta_W}^2 \left(s - m_Z^2\right)} & \left[\ \left(g_V + g_A\right)^2 \mathcal{M}_{LL} + \right. \\ + \left(g_V^2 - g_A^2\right) \mathcal{M}_{LR} + \left(g_V^2 - g_A^2\right) \mathcal{M}_{RL} & \left. + \left(g_V - g_A\right)^2 \mathcal{M}_{RR} \ \right] \end{aligned} \tag{22}$$

Finally, we obtain

$$\begin{aligned} & \mathcal{M}_{Z+\gamma} = \mathcal{M}_Z + \mathcal{M}_\gamma = \\ & = \frac{ie^2}{s} \left[A_{LL}\mathcal{M}_{LL} + A_{LR}\mathcal{M}_{LR} + A_{RL}\mathcal{M}_{RL} + A_{RR}\mathcal{M}_{RR} \right] \end{aligned} \tag{23}$$

with

$$
\begin{aligned}
A_{LL} &= 1 + \frac{s\left(g_V + g_A\right)^2}{4\mathrm{s}_{2\theta_W}^2\left(s - m_Z^2\right)}, \\
A_{LR} = A_{RL} &= 1 + \frac{s\left(g_V^2 - g_A^2\right)}{4\mathrm{s}_{2\theta_W}^2\left(s - m_Z^2\right)}, \\
A_{RR} &= 1 + \frac{s\left(g_V - g_A\right)^2}{4\mathrm{s}_{2\theta_W}^2\left(s - m_Z^2\right)}.
\end{aligned}
\tag{24}
$$

The photon vertex makes no distinction between fermions with different helicities. However, for the Z boson vertex we obtain that it is proportional to $(g_V + g_A)\,\omega_L + (g_V - g_A)\,\omega_R$.
From b) we have :

$$
\mathcal{M}^{ij}_{Z+\gamma} \equiv \frac{ie^2}{s} A_{ij}\mathcal{M}_{ij}. \tag{25}
$$

d) Calculate the sum of the helicity amplitudes squared. Show that this sum can be written as

$$
\sum_{\text{hel}} |\mathcal{M}|^2 = X_1(1 + \cos^2\theta) + X_2\cos\theta
$$

where θ is the μ^- production angle relative to the e^- direction. Express $X_{1,2}$ in terms of A_{ij}.
Sol. [10P] We have

$$
\sum_{\text{hel}} |\mathcal{M}|^2 = \sum_{i,j\in\{L,R\}} |\mathcal{M}^{ij}_{Z+\gamma}|^2 = \frac{e^4}{s^2}\sum_{i,j\in\{L,R\}} A_{ij}^2|\mathcal{M}_{ij}|^2 \tag{26}
$$

In the following we will denote the helicity opposite to that of i, j by $\bar{i}, \bar{j}$, respectively. Below we will make use of : $\omega_i\gamma_0 = \gamma_0\omega_{\bar{i}}$, $\{\gamma_5, \gamma_\mu\} = 0$, $\left(\gamma^0\right)^2 = 1$, $\gamma^0\left(\gamma^\mu\right)^\dagger\gamma^0 = \gamma^\mu$, $\left(\gamma_5\right)^\dagger = \gamma_5$ and $\bar{\psi} = \psi^\dagger\gamma^0$, among other identities.
We have

$$
\begin{aligned}
|\mathcal{M}_{ij}|^2 &= \mathcal{M}_{ij}\mathcal{M}_{ij}^\dagger = \\
&= \bar{v}_2\gamma^\mu\omega_i u_1\bar{u}_3\gamma_\mu\omega_j v_4 \cdot v_4^\dagger\omega_j\left(\gamma_\nu\right)^\dagger\gamma^0 u_3 u_1^\dagger\omega_i\left(\gamma^\nu\right)^\dagger\gamma^0 v_2 = \\
&= \bar{v}_2\gamma^\mu\omega_i u_1\bar{u}_3\gamma_\mu\omega_j v_4\,\underbrace{v_4^\dagger\omega_j\gamma^0}_{=\bar{v}_4\omega_{\bar{j}}}\gamma_\nu u_3\,\underbrace{u_1^\dagger\omega_i\gamma^0}_{=\bar{u}_1\omega_{\bar{i}}}\gamma^\nu v_2 = \\
&= \operatorname{tr}\left[\not{p}_1\omega_{\bar{i}}\gamma^\nu\not{p}_2\gamma^\mu\omega_i\right]\operatorname{tr}\left[\not{p}_3\gamma_\mu\omega_j\not{p}_4\omega_{\bar{j}}\gamma_\nu\right].
\end{aligned}
\tag{27}
$$

Using that $\not{p}\omega_{\bar{i}} = \omega_i \not{p}$, we obtain

$$\begin{aligned}
&|\mathcal{M}_{ij}|^2 = \mathrm{tr}\left[\not{p}_1 \gamma^\nu \not{p}_2 \gamma^\mu \omega_i\right] \mathrm{tr}\left[\not{p}_3 \gamma_\mu \omega_j \not{p}_4 \gamma_\nu\right] = \\
&= p_{1\alpha} p_{2\beta} p_{3\gamma} p_{4\delta} \cdot \\
&\cdot \mathrm{tr}\left[\gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\mu \frac{1}{2}\left(1 \pm_i \gamma_5\right)\right] \mathrm{tr}\left[\gamma^\gamma \gamma_\mu \frac{1}{2}\left(1 \pm_j \gamma_5\right) \gamma^\delta \gamma_\nu\right] = \\
&= p_{1\alpha} p_{2\beta} p_{3\gamma} p_{4\delta} 4 \left(\eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\nu\mu} + \eta^{\alpha\mu}\eta^{\nu\beta} \pm_i i\varepsilon^{\alpha\nu\beta\mu}\right) \cdot \\
&\cdot \left(\eta^\gamma_{\ \mu}\eta^\delta_{\ \nu} - \eta^{\gamma\delta}\eta_{\mu\nu} + \eta^\gamma_{\ \nu}\eta^\delta_\mu \pm_j i\varepsilon^{\gamma\ \delta}_{\ \mu\ \nu}\right).
\end{aligned} \tag{28}$$

The η is a symmetric tensor, while ε is anti-symmetric, therefore all "η-ε" contractions vanish. Therefore, we obtain

$$\begin{aligned}
&|\mathcal{M}_{ij}|^2 = 4 p_{1\alpha} p_{2\beta} p_{3\gamma} p_{4\delta} \left(\ \eta^{\alpha\delta}\eta^{\beta\gamma} - \eta^{\alpha\beta}\eta^{\gamma\delta} + \eta^{\alpha\gamma}\eta^{\beta\delta} - \right. \\
&- \eta^{\alpha\beta}\eta^{\gamma\delta} + 4\eta^{\alpha\beta}\eta^{\gamma\delta} - \eta^{\alpha\beta}\eta^{\gamma\delta} + \\
&\left. + \eta^{\alpha\gamma}\eta^{\beta\delta} - \eta^{\alpha\beta}\eta^{\gamma\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma} - (\pm)_i (\pm)_j \underbrace{\varepsilon^{\alpha\beta\mu\nu}\varepsilon^{\delta\gamma}_{\ \ \mu\nu}}_{=-2\left(\eta^{\alpha\delta}\eta^{\beta\gamma} - \eta^{\alpha\gamma}\eta^{\beta\delta}\right)} \right) = \\
&= 8 p_{1\alpha} p_{2\beta} p_{3\gamma} p_{4\delta} \cdot \\
&\cdot \left[\eta^{\alpha\delta}\eta^{\beta\gamma} + \eta^{\alpha\gamma}\eta^{\beta\delta} + (\pm)_i (\pm)_j \left(\eta^{\alpha\delta}\eta^{\beta\gamma} - \eta^{\alpha\gamma}\eta^{\beta\delta}\right)\right] = \\
&= 8 \Big((p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) + \\
&+ (\pm)_i (\pm)_j \left[(p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)(p_2 \cdot p_4)\right] \Big).
\end{aligned} \tag{29}$$

The Mandelstam variables for $m_e = m_\mu = 0$ are

$$\begin{aligned}
s &= 2p_1 \cdot p_2 = 2p_3 \cdot p_4 \\
t &= -2p_1 \cdot p_3 = -2p_2 \cdot p_4 \\
u &= -2p_1 \cdot p_4 = -2p_2 \cdot p_3
\end{aligned} \tag{30}$$

We therefore obtain

$$\begin{aligned}
|\mathcal{M}_{LL}|^2 = |\mathcal{M}_{RR}|^2 = 16 (p_1 \cdot p_4)(p_2 \cdot p_3) = 4u^2 \\
|\mathcal{M}_{LR}|^2 = |\mathcal{M}_{RL}|^2 = 16 (p_1 \cdot p_3)(p_2 \cdot p_4) = 4t^2
\end{aligned} \tag{31}$$

Therefore

$$\sum_{\mathrm{hel}} |\mathcal{M}|^2 = \frac{4e^4}{s^2}\left[u^2\left(A_{LL}^2 + A_{RR}^2\right) + t^2\left(A_{LR}^2 + A_{RL}^2\right)\right]. \tag{32}$$

In the following we consider the process from the center of momentum frame.

Due to 4-momentum conservation we have : $p_1 + p_2 = p_3 + p_4$.

$$\Rightarrow E_1 = E_2 = E_3 = E_4 \equiv E = \frac{\sqrt{s}}{2}. \tag{33}$$

We furthermore express t and u through s and θ :

$$\begin{aligned} t &= -2p_1 \cdot p_3 = -2\left(E_1 E_3 - |\vec{p}_1||\vec{p}_3|\cos\theta\right) = \\ &= -2E^2\left(1-\cos\theta\right) = -\frac{s}{2}\left(1-\cos\theta\right) \\ u &= -2p_1 \cdot p_4 = -2E^2\left(1-\cos\left(\pi-\theta\right)\right) = -\frac{s}{2}\left(1+\cos\theta\right) \end{aligned} \tag{34}$$

Finally, we obtain

$$\begin{aligned} \sum_{\text{hel}} |\mathcal{M}|^2 &= e^4\left[(1+\cos\theta)^2\left(A_{LL}^2 + A_{RR}^2\right) + \right. \\ &\left. + (1-\cos\theta)^2\left(A_{LR}^2 + A_{RL}^2\right)\right] = X_1\left(1+\cos\theta\right)^2 + X_2\cos\theta \end{aligned} \tag{35}$$

with

$$\begin{aligned} X_1 &= e^4\left(A_{LL}^2 + A_{RR}^2 + A_{LR}^2 + A_{RL}^2\right) \\ X_2 &= 2e^4\left(A_{LL}^2 + A_{RR}^2 - A_{LR}^2 - A_{RL}^2\right) \end{aligned} \tag{36}$$

e) Since the cross-section for $e^+e^- \to \mu^+\mu^-$ is obtained from $\sum|\mathcal{M}|^2$ by integrating over the scattering angle and since

$$\int_{-1}^{1} \mathrm{d}\cos\theta\,\cos\theta = 0,$$

the scattering cross-section is proportional to $X_1(s)$. To study $X_2(s)$, one can define the quantity

$$\mathcal{A} = \frac{\int_0^1 \mathrm{d}\cos\theta\,\mathrm{d}\sigma/\mathrm{d}\cos\theta - \int_{-1}^0 \mathrm{d}\cos\theta\,\mathrm{d}\sigma/\mathrm{d}\cos\theta}{\int_0^1 \mathrm{d}\cos\theta\,\mathrm{d}\sigma/\mathrm{d}\cos\theta + \int_{-1}^0 \mathrm{d}\cos\theta\,\mathrm{d}\sigma/\mathrm{d}\cos\theta},$$

which gives the fractional difference in the number of negatively charged muons which are produced in the forward and backward hemispheres, defined w.r.t. the electron direction of motion. Calculate the above quantity in terms of $X_{1,2}$.

Sol. [2P] The differential cross section for $e^-e^+ \to \mu^-\mu^+$ reads

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\frac{1}{4}\sum_{\text{hel}}|\mathcal{M}|^2}{64\pi^2 s}, \tag{37}$$

where the $\frac{1}{4}$ factor comes from averaging over initial helicities.
The total cross section is given by

$$\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = 2\pi\int_{-1}^{1}\mathrm{d}c_\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}c_\theta}. \tag{38}$$

We define

$$\begin{aligned} F &\equiv 2\pi\int_0^1 \mathrm{d}c_\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}c_\theta} = \frac{2}{4^4\pi s}\int_0^1 \mathrm{d}z\,\left(X_1\left(1+z^2\right)+X_2 z\right) = \\ &= \frac{2}{4^4\pi s}\left(\frac{4}{3}X_1+\frac{1}{2}X_2\right), \\ B &\equiv 2\pi\int_{-1}^0 \mathrm{d}c_\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}c_\theta} = \frac{2}{4^4\pi s}\int_{-1}^0 \mathrm{d}z\,\left(X_1\left(1+z^2\right)+X_2 z\right) = \\ &= \frac{2}{4^4\pi s}\left(\frac{4}{3}X_1-\frac{1}{2}X_2\right). \end{aligned} \tag{39}$$

Therefore

$$\mathcal{A} = \frac{F-B}{F+B} = \frac{3}{8}\frac{X_2}{X_1}. \tag{40}$$

f) Find $\mathcal{A}$ in the small energy limit $s \ll m_Z^2$ and in the Z-resonance limit $s \to m_Z^2$, where the photon exchange can, effectively, be neglected.
Sol. [1P]
From e) we have :

$$\mathcal{A} = \frac{3sg_A^2\csc^2(2\theta_W)\left[2s-2m_Z^2+sg_V^2\csc^2(2\theta_W)\right]}{16(s-m_Z^2)^2+8g_V^2 s(s-m_Z^2)\csc^2(2\theta_W)+(g_V^2+g_A^2)^2 s^2\csc^4(2\theta_W)}. \tag{41}$$

In the limit $s \ll m_Z^2$, i.e. for $\frac{s}{m_Z^2} \ll 1$, we have

$$\mathcal{A}(s \ll m_Z^2) = -\frac{3}{8}\frac{g_A^2}{\sin^2(2\theta_W)}\frac{s}{m_Z^2} + \mathcal{O}\left(\left[\frac{s}{m_Z^2}\right]^2\right) < 0. \tag{42}$$

Whereas in the limit $s \to m_Z^2$ we have

$$\mathcal{A}(s \to m_Z^2) = \frac{3g_V^2 g_A^2}{(g_V^2+g_A^2)^2} > 0. \tag{43}$$

Problem 5 (17 points)

Consider the leptonic Yukawa sector of the SM with only two generations and including the right-handed neutrinos.

a) Write down the most general, renormalizable mass terms for the leptons that are compatible with the SM symmetries.

Sol. [2P]

The most general mass term comes from the Yukawa-Interaction

$$-Y_{ij}^{(e)} \bar{E}_L^i H e_R^j - Y_{ij}^{(\nu)} \bar{E}_L^i i\sigma_2 H^* \nu_R^j + \text{h.c.},$$

which after SSB takes the form

$$\mathcal{L}_{\text{Dirac}} = -M_{ij}^{(e)} \bar{e}_L^i e_R^j - M_{ij}^{(\nu)} \bar{\nu}_L^i \nu_R^j,$$

where $i, j \in \{1, 2\}$ run over lepton generations, and includes the Majorana mass term

$$\mathcal{L}_{\text{Majorana}} = -\frac{1}{2} M_{ij}^{(R)} \nu_R^{T,i} C \nu_R^j.$$

b) Now take the Dirac mass matrix of the neutrinos to be diagonal and the electron/muon Dirac mass matrix to have the following form (in the basis of weak interaction eigenstates)

$$M^{(e)} = m \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} .$$

Here, m is a parameter with dimensions of mass, and a, b complex. In the following, assume that the Dirac masses are the only masses in the Yukawa sector. Now go to the mass state eigenbasis and find the flavor mixing matrix V that appears in the interaction between the mass eigenstates and the gauge fields. *Hint : You may use the fact that the diagonalization preserves the determinant and the trace of a matrix.*

Sol. [5P]

Since $M_{ij}^{(\nu)}$ is already diagonal, we only have to diagonalize $M_{ij}^{(e)}$. Since it is hermitean, we only need the unitary matrix L_e. Hence the PMNS-matrix (analog of the CKM-matrix) will be given by

$$V = L_e.$$

To find L_e, we note that the eigenvalues m_1, m_2 satisfy

$$m_1 m_2 = \det M^{(e)} = 0$$
$$m_1 + m_2 = \text{tr} M^{(e)} = mr^2 ,$$

where we defined $r \equiv \sqrt{|a|^2 + |b|^2}$. So the eigenvalues are given by

$$m_1 = 0\,, \quad m_2 = mr^2\,,$$

and the (normalized) eigenvectors are

$$\vec{v}_1 = \frac{1}{r}\begin{pmatrix} b^* \\ -a^* \end{pmatrix}\,, \quad \vec{v}_2 = \frac{1}{r}\begin{pmatrix} a \\ b \end{pmatrix}$$

Then, the PMNS-matrix (up to non-physical phases) is

$$V = \begin{pmatrix} \cos\theta e^{-i\chi} & \sin\theta e^{i\varphi} \\ -\sin\theta e^{-i\varphi} & \cos\theta e^{i\chi} \end{pmatrix}\,,$$

where we defined $\sin\theta = |a|/r$ and $\cos\theta = |b|/r$.

c) Reintroduce the most general mass term from part a) and consider an arbitrary Dirac mass matrix. Without an explicit calculation, argue how many independent and physical parameters the new matrix V will have.

Sol. [5P]

When the neutrinos have both a Dirac and a Majorana mass term, then they are in general Majorana fermions. Therefore, we cannot use phase re-definitions (of the neutrinos) in order to rotate away phases in the PMNS-matrix. Hence, we can only use the (left-handed) electrons/muons to remove 2 phases. So, in general, V will have 1 angle and 3-2=1 phase.

d) Will there be a physical CP-violating phase?

Sol. [1P] Yes.

e) Finally, consider a situation similar to part b), but with 3 generations of leptons (again, consider only Dirac masses). The neutrino mass matrix is also diagonal and the electron-like lepton mass matrix has the form (in the basis of weak interaction eigenstates)

$$M^{(e)} = m\begin{pmatrix} |a|^2 & ab^* & 0 \\ a^*b & |b|^2 & 0 \\ 0 & 0 & |c|^2 \end{pmatrix}\,,$$

where m is a parameter with dimensions of mass, and a, b, c complex. How many independent parameters will the matrix V have now? Will there be a physical CP-violating phase? Explain.

Sol. [4P]

Now the PMNS-matrix is given by

$$V = \begin{pmatrix} \cos\theta e^{-i\chi} & \sin\theta e^{i\varphi} & 0 \\ -\sin\theta e^{-i\varphi} & \cos\theta e^{i\chi} & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

so it has 1 angle and 2 phases. However, we can see explicitly that the phases can be removed by the lepton phase re-definitions by writing

$$V = \begin{pmatrix} e^{i\varphi} & 0 & 0 \\ 0 & e^{i\chi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i(\chi+\varphi)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, there is no CP-violating phase.

Problem 6 (8 points)

a) Draw the tree-level Feynman diagrams for the following decays

$$D^0 \to K^-\pi^+,$$
$$D^0 \to K^+\pi^-,$$

with $D^0 = |c\bar{u}\rangle\,, K^- = |s\bar{u}\rangle\,, K^+ = |\bar{s}u\rangle\,, \pi^+ = |u\bar{d}\rangle\,, \pi^- = |\bar{u}d\rangle$. These are mediated by the W boson.

Sol. [4P]

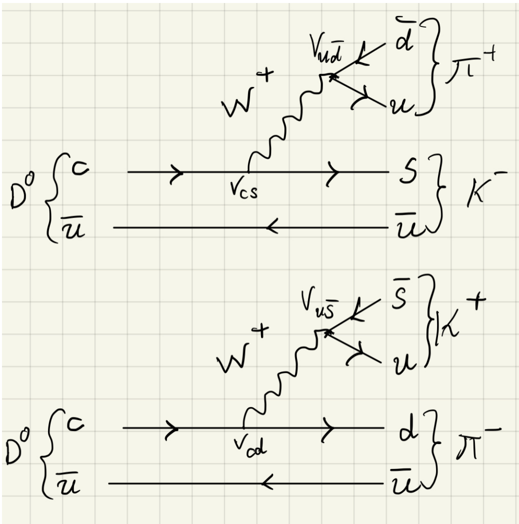

FIGURE 1 – Tree-level Feynman diagrams

b) Estimate the ratio of the two decay rates

$$\frac{\Gamma(D^0 \to K^-\pi^+)}{\Gamma(D^0 \to K^+\pi^-)}$$

Use the following approximation for the CKM matrix

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$
$$= \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4),$$

with $\lambda = \sin\theta_C$ (θ_C is the Cabibbo angle).
Sol. [4P]

$$\begin{aligned}
\frac{\Gamma(D^0 \to K^-\pi^+)}{\Gamma(D^0 \to K^+\pi^-)} &= \left|\frac{V_{ud}V_{cs}}{V_{us}V_{cd}}\right|^2 \\
&\sim \left|\frac{(1-\lambda^2/2)^2}{\lambda^2}\right|^2 \\
&\sim \left(\frac{1}{\lambda} - \frac{\lambda}{2}\right)^4 \\
&\sim 380
\end{aligned}$$

In PDG $R^{-1} = (3.37 \pm 0.21) \times 10^-3 \sim 1/300$, page 54 `https://pdg.lbl.gov/2008/listings/s032.pdf` and `https://pdglive.lbl.gov/BranchingRatio.action?desig=50&parCode=S032&home=MXXX035`

Problem 7 (16 points)

Consider a Higgs theory of a scalar field ϕ transforming under two different abelian gauge symmetries $U(1)$ and $U(1)'$ with gauge fields A_μ and A'_μ and gauge couplings g and g', respectively. Assume that the charges of ϕ under the two symmetries are q and q'. Assume that the absolute value of ϕ has a non-zero vacuum expectation value $\langle|\phi|\rangle = v$. Write down the mass matrix for the gauge fields.

a) Which combination of the two gauge fields remains massless ?

b) Which combination gets a mass as a result of the Higgs effect ? What is the value of the mass ?

Sol. [16P]

Let's first write down the mass matrix for the gauge field. We star from the lagrangian

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - V(\phi),$$

where the covariant derivative is given by

$$D_\mu\phi = \partial_\mu\phi - iqgA_\mu\phi - iq'g'A'_\mu\phi.$$

The mass term for the gauge bosons reads as :

$$\begin{aligned}\mathcal{L}_M &= (qgA_\mu + q'g'A'_\mu)(qgA^\mu + q'g'A'^\mu)v^2 \\ &= \frac{1}{2}(A_\mu, A'_\mu)M^2 \begin{pmatrix} A^\mu \\ A'^\mu \end{pmatrix},\end{aligned}$$

where M^2 is the mass matrix

$$M^2 = 2v^2 \begin{pmatrix} q^2g^2 & qq'gg' \\ qq'gg' & q'^2g'^2 \end{pmatrix}.$$

Lets diagonalize M^2.

$$\begin{aligned}\det M^2 &= 0 \\ \mathrm{tr} M^2 &= 2v^2(q^2g^2 + q'^2g'^2) \equiv m^2.\end{aligned}$$

As expected one gauge boson remains massless and one gets a mass $m = \sqrt{2(q^2g^2 + q'^2g'^2)}v$. The corresponding (normalized) eigenvectors are :

$$\frac{1}{\sqrt{q^2g^2 + q'^2g'^2}} \begin{pmatrix} q'g' \\ -qg \end{pmatrix},$$
$$\frac{1}{\sqrt{q^2g^2 + q'^2g'^2}} \begin{pmatrix} qg \\ q'g' \end{pmatrix}$$

From here, it follows that the massless combination is

$$B_\mu = \frac{q'g'A_\mu - qgA'_\mu}{\sqrt{q^2g^2 + q'^2g'^2}},$$

while the massive combination is

$$B'_\mu = \frac{qgA_\mu + q'g'A'_\mu}{\sqrt{q^2g^2 + q'^2g'^2}}$$