Ludwig-Maximilians-Universität München

QCD and Standard Model

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Guidelines :

- The exam consists of 7 problems.
- The duration of the exam is 24 hours.
- Please write your name or matriculation number on every sheet that you hand in.
- Your answers should be comprehensible and readable.

GOOD LUCK !

Problem 1	14 P
Problem 2	12 P
Problem 3	12 P
Problem 4	21 P
Problem 5	17 P
Problem 6	8 P
Problem 7	16 P

Total	100 P

Problem 1 (14 points)

Consider a theory invariant under a global SU(2) symmetry with two scalar fields, Σ and Φ, each transforming in the adjoint representation of the group, i.e.

$$\Sigma \to \Sigma' = U\Sigma U^\dagger \ ,$$
$$\Phi \to \Phi' = U\Phi U^\dagger \ ,$$

where $U \in SU(2)$ and $\dagger$ stands for Hermitian conjugation.

a) Assume that the scalar fields do not interact with each other. Then the most general potential for each scalar field could involve the invariants

   i) $\mathrm{tr}Z$,
   ii) $\mathrm{tr}Z^2$,
   iii) $\mathrm{tr}Z^3$,
   iv) $\mathrm{tr}Z^4$,
   v) $(\mathrm{tr}Z^2)^2$,
   vi) $\det(Z)$,
   vii) $(\det(Z))^2$,

   with $Z = \Sigma, \Phi$. Which of the above invariants are identically zero? How many of non-zero invariants are independent? (*Hint : Bring the fields into a simple form by a group transformation.*)
   Use this, to write down the Lagrangian for Σ and Φ.

b) Arrange the potential such that the vacuum expectation value (vev) of each scalar field is non-zero. Find the vev by explicitly minimizing the potential.

c) How many generators are broken? Does the answer depend on the "direction" of the vacuum?

d) Assume now that the global SU(2) is made local. Construct the appropriate covariant derivatives for Σ and Φ.

e) Compute the masses of the gauge bosons on top of the vacua you found in b). Is it possible to find that all gauge bosons have different masses? Explain.

Problem 2 (12 points)

Consider a theory invariant under a global SU(N), $N > 4$ symmetry. Take X to be a symmetric $N \times N$ matrix scalar field that under an SU(N) transformation behaves as

$$X \to X' = U X U^T \ ,$$

where $U \in SU(N)$ and T stands for transpose.

a) Write down the most general, renormalizable, SU(N)- and Lorentz-invariant Lagrangian for X in four spacetime dimensions.

b) Assume now that we gauge the SU(N) symmetry. Construct the appropriate covariant derivative for X. If you just "guess" it without a derivation, you have to show that it indeed transforms as a covariant derivative.

Problem 3 (12 points)

Focus only on the leptonic sector of the Standard Model supplemented with three right-handed neutrinos.

a) Take the Yukawa couplings to be zero. What are the global symmetries (apart from the gauged $SU(2) \times U(1)$) ?

b) What are the symmetries once we switch on arbitrary Yukawa couplings ?

c) What is the symmetry if we assume that the Yukawa couplings are non-zero diagonal matrices ?

d) Do your answers for parts a) and b) change if the right-handed neutrinos are removed ? Explain.

Problem 4 (21 points)

Consider the process

$$e^-(p_1)e^+(p_2) \to \mu^-(p_3)\mu^+(p_4) \ ,$$

and assume that all fermion masses are zero.

In the Standard Model, at tree-level, this can occur through the exchange of a photon or Z boson (you can neglect the Higgs boson exchange in the zero-fermion-mass limit). The interaction of charged leptons with photons is given by the vertex $ie\gamma_\mu$, while their interaction with Z-bosons is given by $ie/(2\sin(2\theta_W))(g_V\gamma_\mu + g_A\gamma_\mu\gamma_5)$.

a) Starting from the (covariant) kinetic terms of the charged leptons, show that $g_V = (1 - 4\sin^2\theta_W)$ and $g_A = 1$.

b) Split the above vertices into vertices that describe interactions of vector bosons with fermions of definite helicities. Show that the amplitude $e^+e^- \to \mu^+\mu^-$ can be written as

$$\begin{aligned}\mathcal{M}_{Z+\gamma} \equiv \mathcal{M}_Z + \mathcal{M}_\gamma = \frac{ie^2}{s}\,[&A_{LL}\bar{v}_{p_2}\gamma^\mu\omega_L u(p_1)\bar{u}(p_3)\gamma_\mu\omega_L v(p_4)+\\ &+A_{RL}\bar{v}_{p_2}\gamma^\mu\omega_R u(p_1)\bar{u}(p_3)\gamma_\mu\omega_L v(p_4)+\\ &+A_{LR}\bar{v}_{p_2}\gamma^\mu\omega_L u(p_1)\bar{u}(p_3)\gamma_\mu\omega_R v(p_4)+\\ &+A_{RR}\bar{v}_{p_2}\gamma^\mu\omega_R u(p_1)\bar{u}(p_3)\gamma_\mu\omega_R v(p_4)\,]\end{aligned}$$

where $\omega_{L/R} = P_{L/R} = (1 \pm \gamma_5)/2$ are projectors on different helicity states and s is the square of the center-of-mass energy. Express the coefficients A_{ij} through the couplings $g_{A,V}$.

c) Calculate all the relevant helicity amplitudes in terms of spinor products.

d) Calculate the sum of the helicity amplitudes squared. Show that this sum can be written as

$$\sum_{\text{hel}} |\mathcal{M}|^2 = X_1(1 + \cos^2\theta) + X_2\cos\theta$$

where θ is the μ^- production angle relative to the e^- direction. Express $X_{1,2}$ in terms of A_{ij}.

e) Since the cross-section for $e^+e^- \to \mu^+\mu^-$ is obtained from $\sum |\mathcal{M}|^2$ by integrating over the scattering angle and since

$$\int_{-1}^{1} \mathrm{d}\cos\theta\,\cos\theta = 0,$$

the scattering cross-section is proportional to $X_1(s)$. To study $X_2(s)$, one can define the quantity

$$\mathcal{A} = \frac{\int_0^1 \mathrm{d}\cos\theta\,\mathrm{d}\sigma/\mathrm{d}\cos\theta - \int_{-1}^0 \mathrm{d}\cos\theta\,\mathrm{d}\sigma/\mathrm{d}\cos\theta}{\int_0^1 \mathrm{d}\cos\theta\,\mathrm{d}\sigma/\mathrm{d}\cos\theta + \int_{-1}^0 \mathrm{d}\cos\theta\,\mathrm{d}\sigma/\mathrm{d}\cos\theta},$$

which gives the fractional difference in the number of negatively charged muons which are produced in the forward and backward hemispheres, defined w.r.t. the electron direction of motion. Calculate the above quantity in terms of $X_{1,2}$.

f) Find $\mathcal{A}$ in the small energy limit $s \ll M_Z^2$ and in the Z-resonance limit $s \to M_Z^2$, where the photon exchange can, effectively, be neglected.

Problem 5 (17 points)

Consider the leptonic Yukawa sector of the SM with only two generations and including the right-handed neutrinos.

a) Write down the most general, renormalizable mass terms for the leptons that are compatible with the SM symmetries.

b) Now take the Dirac mass matrix of the neutrinos to be diagonal and the electron/muon Dirac mass matrix to have the following form (in the basis of weak interaction eigenstates)

$$M^{(e)} = m \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} .$$

Here, m is a parameter with dimensions of mass, and a, b complex. In the following, assume that the Dirac masses are the only masses in the Yukawa sector. Now go to the mass state eigenbasis and find the flavor mixing matrix V that appears in the interaction between the mass eigenstates and the gauge fields. *Hint : You may use the fact that the diagonalization preserves the determinant and the trace of a matrix.*

c) Reintroduce the most general mass term from part a) and consider an arbitrary Dirac mass matrix. Without an explicit calculation, argue how many independent and physical parameters the new matrix V will have.

d) Will there be a physical CP-violating phase ?

e) Finally, consider a situation similar to part b), but with 3 generations of leptons (again, consider only Dirac masses). The neutrino mass matrix is also diagonal and the electron-like lepton mass matrix has the form (in the basis of weak interaction eigenstates)

$$M^{(e)} = m \begin{pmatrix} |a|^2 & ab^* & 0 \\ a^*b & |b|^2 & 0 \\ 0 & 0 & |c|^2 \end{pmatrix} ,$$

where m is a parameter with dimensions of mass, and a, b, c complex. How many independent parameters will the matrix V have now? Will there be a physical CP-violating phase? Explain.

Problem 6 (8 points)

a) Draw the tree-level Feynman diagrams for the following decays

$$D^0 \to K^-\pi^+,$$
$$D^0 \to K^+\pi^-,$$

with $D^0 = |c\bar{u}\rangle\,, K^- = |s\bar{u}\rangle\,, K^+ = |\bar{s}u\rangle\,, \pi^+ = |u\bar{d}\rangle\,, \pi^- = |\bar{u}d\rangle$. These are mediated by the W boson.

b) Estimate the ratio of the two decay rates

$$\frac{\Gamma(D^0 \to K^-\pi^+)}{\Gamma(D^0 \to K^+\pi^-)}$$

Use the following approximation for the CKM matrix

$$\begin{aligned} V_{CKM} &= \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \\ &= \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4), \end{aligned}$$

with $\lambda = \sin\theta_C$ (θ_C is the Cabibbo angle).

Problem 7 (16 points)

Consider a Higgs theory of a scalar field ϕ transforming under two different abelian gauge symmetries $U(1)$ and $U(1)'$ with gauge fields A_μ and A'_μ and gauge couplings g and g', respectively. Assume that the charges of ϕ under the two symmetries are q and q'. Assume that the absolute value of ϕ has a non-zero vacuum expectation value $\langle|\phi|\rangle = v$. Write down the mass matrix for the gauge fields.

a) Which combination of the two gauge fields remains massless?

b) Which combination gets a mass as a result of the Higgs effect? What is the value of the mass?