a) · let's focus on the Yukawa interaction terms for the quarks

$$\mathcal{L}_Y^{(q)} = -\Lambda_{ij}^{(d)} \bar{Q}_L^i H d_R^j - \Lambda_{ij}^{(u)} \bar{Q}_L^i \tilde{H} u_R^j + h.c.$$

· note that the Yukawa matrices $\Lambda_{ij}^{(d)}$ and $\Lambda_{ij}^{(u)}$ are general complex matrices.

· after symmetry breaking the ~~yea~~ quarks acquire masses

$$-\Lambda_{ij}^{(d)} \bar{Q}_L^i \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} d_R^j - \Lambda_{ij}^{(u)} \bar{Q}_L^i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} u_R^j + h.c.$$

$$= -\frac{v}{\sqrt{2}} \Big(\underbrace{\Lambda_{ij}^{(d)} \bar{d}_L^i d_R^j}_{\equiv \bar{d}_L \Lambda_d d_R} + \underbrace{\Lambda_{ij}^{(u)} \bar{u}_L^i u_R^j}_{\equiv \bar{u}_L \Lambda_u u_R} + h.c. \Big)$$

↳ since $\Lambda_{d,u} \Lambda_{d,u}^\dagger$ is hermitean, we can always find a diagonal matrix $M_{d,u}$ such that

$\Lambda_{d,u} \Lambda_{d,u}^\dagger = U_{d,u} M_{d,u}^2 U_{d,u}^\dagger$, with the unitary matrices $U_{d,u}$.

↳ $\Lambda_{d,u}$ itself does not have to be hermitean, however we can generically write $\Lambda_{d,u} = U_{d,u} M_{d,u} K_{d,u}^\dagger$, with another unitary matrix $K_{d,u}$.

$\Rightarrow \mathcal{L}_Y^{(q)} \supset -\frac{v}{\sqrt{2}} \left(\bar{d}_L U_d M_d K_d^\dagger d_R + \bar{u}_L U_u M_u K_u^\dagger u_R \right)$

$+ h.c.$

$\rightarrow$ so we can rotate $d_R \rightarrow K_d d_R$, $u_R \rightarrow K_u u_R$

and $d_L \rightarrow U_d d_L$ $\left(\Rightarrow \bar{d}_L \rightarrow \bar{d}_L U_d^\dagger \right)$,

$u_L \rightarrow U_u u_L$

to get (in the mass basis)

$\mathcal{L}_Y^{(q)} \supset - \left(m_i^{(d)} \bar{d}_L^i d_R^i + m_i^{(u)} \bar{u}_L^i u_R^i \right) + h.c.$

where $m_i^{(d)} = \frac{v}{\sqrt{2}} M_{d,ii}$, $m_i^{(u)} = \frac{v}{\sqrt{2}} M_{u,ii}$

i-th eigenvalue of $M_{d,u}$

- now, how does this change of basis affect the quark-gauge field-interactions?

$\rightarrow$ $D_\mu \psi_j = \left(\partial_\mu - i g W_\mu^a \tau^a - i g' \frac{Y}{2} B_\mu \right) \psi_j$

$= \left[\partial_\mu - \frac{i}{2} \begin{pmatrix} g W_\mu^3 + g' Y B_\mu & g(W_\mu^1 - i W_\mu^2) \\ g(W_\mu^1 + i W_\mu^2) & -g W_\mu^3 + g' Y B_\mu \end{pmatrix} \right] \psi_j$

$$\hookrightarrow \quad i\sum_j \bar{\psi}_j \not{D} \psi_j \supset i\bar{Q}_L \not{D} Q_L + i\bar{u}_R \not{D} u_R$$

$$+ i\bar{d}_R \not{D} d_R$$

$$= (\bar{u}_L^i, \bar{d}_L^i) \left[i\not{\partial} + \gamma^\mu \begin{pmatrix} \frac{g}{2} W_\mu^3 + \frac{g'}{6} B_\mu & \frac{g}{2}(W_\mu^1 - iW_\mu^2) \\ \frac{g}{2}(W_\mu^1 + iW_\mu^2) & -\frac{g}{2} W_\mu^3 + \frac{g'}{6} B_\mu \end{pmatrix} \right] \times$$

$$\times \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix}$$

$$+ \bar{u}_R^i \left(i\not{\partial} + \frac{2}{3} g' \not{B} \right) u_R^i$$

$$+ \bar{d}_R^i \left(i\not{\partial} - \frac{1}{3} g' \not{B} \right) d_R^i$$

where we used $Q = T^3 + \frac{1}{2} Y$ $\quad \rightarrow \quad Y(Q_L) = \frac{1}{3}, \; Y(u_R) = \frac{4}{3},$

$Y(d_R) = -\frac{2}{3}$

$\hookrightarrow$ note that in the original basis (flavour basis) the interaction terms don't mix generations.

$\hookrightarrow$ furthermore, the hypercharge interactions don't mix flavours, so they are invariant under the change of basis $u_R \to K_u u_R$, $d_R \to K_d u_R$.

↳ the same is true for the couplings to W_μ^3 and B_μ (since they come from the diagonal part)

↳ however, the interaction terms involving $W_\mu^\pm$ will lead to

$$\frac{g}{2}\left[\bar{u}_L^i \gamma^\mu W_\mu^+ d_L^i + \bar{d}_L^i \gamma^\mu W_\mu^- u_L^i\right]$$

$$\xrightarrow[\text{mass basis}]{\text{going to}} \frac{g}{2}\Big[W_\mu^+ \bar{u}_L^i \gamma^\mu \underbrace{\left(U_u^+ U_d\right)}_{\equiv V}{}_{ij}\, d_L^j$$

$$+ W_\mu^- \bar{d}_L^i \gamma^\mu \underbrace{\left(U_d^+ U_u\right)}_{\equiv V^+}{}_{ij}\, u_L^j\Big]$$

↳ V_{ij} is known as the CKM-matrix

b) . V is obviously unitary, so it can be written as $V = e^{iA}$, with $A^+ = A$

↳ since A is $N \times N$ hermitian matrix, it has N^2 real dof's

↳ if V were real, it would be an ~~...~~ $N \times N$ orthogonal matrix and hence have $\frac{N(N-1)}{2}$ real dof's (number of independent rotations of $O(N)$)

↳ so V has $\frac{N(N-1)}{2}$ angles and

$N^2 - \frac{N(N-1)}{2} = \frac{N(N+1)}{2}$ phases.

↳ note that in the Yukawa sector there are $2N$ residual global $U(1)$ symmetries, since we can transform $d_L^j \to e^{i\alpha_j} d_L^j$, $d_R^j \to e^{i\alpha_j} d_R^j$ and $u_L^j \to e^{i\beta_j} u_L^j$, $u_R^j \to e^{i\beta_j} u_R^j$, with $j \in [1,2,...,N]$ leaving the mass terms invariant.
So we can use these quark field redefinitions to remove complex phases of V.
However, if all rotations are the same, $\alpha_j = \beta_j = \theta$, then V is unchanged, hence we can only remove $2N-1$ phases in this way.

↳ so there are $\frac{N(N+1)}{2} - (2N-1) = \frac{N^2-3N+2}{2}$ independent phases.

· for $N=3$: 3 angles, 1 phase
$N=2$: 1 angle (Cabibbo angle)

c) · let's look at the up-type quark mass terms

$$\mathcal{L}_Y^{(q)} \supset -\frac{v}{\sqrt{2}}\left(\underbrace{\bar{u}_L}_{=\bar{u}P_R} \lambda_u \underbrace{u_R}_{= P_R u} + \underbrace{\bar{u}_R}_{=\bar{u}P_L} \lambda_u^\dagger \underbrace{u_L}_{=P_L u}\right)$$

$$= -\frac{v}{2\sqrt{2}}\left[\bar{u}(\lambda_u + \lambda_u^\dagger)u + \bar{u}(\lambda_u - \lambda_u^\dagger)\gamma^5 u\right]$$

where we used $P_{L/R} = \frac{1}{2}(1 \mp \gamma^5)$.

↳ how does this transform under CP?

· $u_i \xrightarrow{P} \gamma^0 u_i$, $u_i \xrightarrow{C} C\bar{u}_i^T$

$\Rightarrow u_i \xrightarrow{CP} \gamma^0 C \bar{u}^T$

↳ $\bar{u}_i \xrightarrow{CP} u_i^T C\gamma^0$

$\Rightarrow \bar{u}_i u_j \xrightarrow{CP} \bar{u}_j u_i$

$\bar{u}_i \gamma^5 u_j \xrightarrow{CP} -\bar{u}_j \gamma^5 u_i$

↳ so the Yukawa term becomes

$$\xrightarrow{CP} -\frac{v}{2\sqrt{2}}\left[\bar{u}(\lambda_u + \lambda_u^\dagger)^T u - \bar{u}(\lambda_u - \lambda_u^\dagger)^T \gamma^5 u\right] =$$

$$= -\frac{v}{2\sqrt{2}} \left[\bar{u} \left(\Lambda_u^* + (\Lambda_u^\dagger)^* \right) u + \bar{u} \left(\Lambda_u^* - (\Lambda_u^\dagger)^* \right) \gamma^5 u \right]$$

↳ note that $\bar{u}_i u_j$ and $\bar{u}_i \gamma^5 u_j$ are invariant under T (time reversal), so if the coefficents $(\Lambda_u^*)_{ij}$ etc. are complex, the above expression is not invariant under T, since $i \xrightarrow{T} -i$. But since CPT has to be a symmetry, this would imply that CP has to be violated.

↳ so the Yukawa sector of the quarks is only CP invariant, if $\Lambda_{u,d}^* = \Lambda_{u,d}$.

- note that this is not a basis-invariant statement. However, one can show that if the CKM-matrix V is real, there is no CP-violation. So the experimental evidence of CP violation implied that V cannot be real and hence the number of quark generations $N > 2$.

d) · for massless neutrinos there is no Yukawa term in the lepton sector like

$$- \Lambda_{ij}^{(\nu)} \bar{E}_L^i \tilde{H} \nu_R^j + h.c.$$

(analogous to the up-like quark in the quark sector).

So in order to diagonalize $\Lambda_{ij}^{(e)} \bar{E}_L^i H e_R^j$, we just need two unitary matrices K_e and U_e to go to the mass basis via

$$e_R \to K_e e_R \quad , \quad e_L \to U_e e_L$$

· the lepton-gauge field-interactions have the form

$$i \bar{E}_L \not{D} E_L + i \bar{e}_R \not{D} e_R$$

$$= (\bar{\nu}_L^i, \bar{e}_L^i) \left[i\not{\partial} + \frac{1}{2}\gamma^\mu \begin{pmatrix} g W_\mu^3 - g' B_\mu & g W_\mu^+ \\ g W_\mu^- & -g W_\mu^3 - g' B_\mu \end{pmatrix} \right] \begin{pmatrix} \nu_L^i \\ e_L^i \end{pmatrix}$$

$$+ \bar{e}_R^i (i\not{\partial} - g' \not{B}) e_R$$

↳ now let's look at the flavour mixing part

$$\frac{g}{2}\left[\bar{\nu}_L^i \not{W}^+ e_L^i + \bar{e}_L^i \not{W}^- \nu_L^i\right]$$

↳ going to the mass-basis by $e_L \to U_e e_L$ can now be undone by the rotation $\nu_L \to U_e \nu_L$, which can be done with impunity, because there is no corresponding mass term in the Yukawa sector.

↳ hence for massless neutrinos, the lepton sector can be diagonal in the mass terms and the interaction terms at the same time!

a) You saw in the lecture that there are two different ways to write down a Lorentz invariant mass term for fermions.

Dirac mass: $m_D \bar{\psi}_L \psi_R + h.c.$

Majorana mass: $m_M \bar{\psi}^c_L \psi_L + h.c.$

(or $m_M \bar{\psi}^c_R \psi_R + h.c.$)

with $\psi^c_L = C \bar{\psi}_L^T = i\gamma^2 \psi^*_L \equiv (\psi^c)_R$

$\hookrightarrow m_M \bar{\psi}^c_L \psi_L = m_M \psi_L^T C \psi_L$

- the Majorana mass term cannot describe charged particles, since it violates the conservation of charge
  $\hookrightarrow$ so the neutrino is the only SM fermion, which could, in principle, have a Dirac mass and/or a Majorana mass. Note that in the latter case, the neutrino cannot carry a lepton number, so lepton number would be violated in the SM!

b) . for the Dirac mass we can use, analogously to the quark case, the Higgs doublet with opposite hypercharge, $\tilde{H} = i\sigma^2 H^*$:

Dirac: $-\lambda_{ij}^{(\nu)} \bar{E}_L^i \tilde{H} \nu_R^j + h.c.$

Majorana: $-M_{ij} \bar{\nu}_R^{c\,i} \nu_R^j + h.c.$

↳ note that we cannot write down a Majorana mass term for ν_L, because it transforms under $SU(2)$ ~~[illegible]~~ $\times U(1)_Y$ (it carries quantum numbers).

c) . let's first neglect the Majorana mass term.

. we can now again (like in ex. 4.2.) rotate to the mass basis via

$e_R \to K_e e_R$, $e_L \to U_e e_L$, $\nu_R \to K_\nu \nu_R$,

$\nu_L \to U_\nu \nu_L$

↳ now the mass terms of the leptons are diagonal, however the (charged) interaction term

$$\frac{g}{\sqrt{2}}\left[\bar{\nu}_L^i \not{W}^+ e_L^i + \bar{e}_L^i \not{W}^- \nu_L^i\right]$$ becomes

$$\xrightarrow[\text{basis}]{\text{mass}} \frac{g}{\sqrt{2}}\left[\underbrace{(U_\nu^\dagger U_e)_{ij}}_{=P} \bar{\nu}_L^i \not{W}^+ e_L^j + \underbrace{(U_e^\dagger U_\nu)_{ij}}_{\equiv P^\dagger} \bar{e}_L^i \not{W}^- \nu_L^j\right]$$

where $P_{ij} = (U_\nu^\dagger U_e)_{ij}$ is the PMNS-matrix

d) · let's denote the neutrino flavors (weak basis) by ν_α and the mass eigenstates by ν_i , $\alpha = e, \mu, \tau$, $i = 1, 2, 3$

$\Rightarrow \nu_\alpha = P_{\alpha i} \nu_i$ (sum over i)

· let's consider a beam of neutrinos, which at time $t=0$ get produced as a weak eigenstate $|\nu_\alpha(0)\rangle$. We can now expand this in mass eigenstates as $|\nu_\alpha(0)\rangle = P_{\alpha i}|\nu_i\rangle$

· let's assume that all neutrinos in the beam have the same momentum, then $E_i^2 = \vec{p}^{\,2} + m_i^2$

· now the different energy eigenvalues E_i of the energy eigenstates $|\nu_i\rangle$ will cause the state $|\nu_\alpha(0)\rangle$ to evolve:

$$|\nu_\alpha(t)\rangle = e^{-iHt}|\nu_\alpha(0)\rangle = P_{\alpha i}\, e^{-iE_i t}|\nu_i\rangle$$

↳ so the probability of finding ν_β in a beam of (initially) ν_α after time t is

$$P_{\nu_\alpha \to \nu_\beta}(t) = |\langle \nu_\beta(0)|\nu_\alpha(t)\rangle|^2$$

$$= \left|\langle \nu_j|\, P^*_{\beta j}\, P_{\alpha i}\, e^{-iE_i t}|\nu_i\rangle\right|^2$$

$$= \left| P^*_{\beta i}\, P_{\alpha i}\, e^{-iE_i t}\right|^2$$

$$= \sum_i |P_{\alpha i}|^2 |P_{\beta i}|^2 + \sum_{i\neq j} P_{\alpha i} P^*_{\beta i} P^*_{\alpha j} P_{\beta j} \cdot e^{-i(E_i - E_j)t}$$

· for ultra-relativistic neutrinos $(\underbrace{\vec{p}^{\,2}}_{\equiv p^2} \gg m_i^2)$,

$$E_i - E_j \simeq \frac{m_i^2 - m_j^2}{2p}$$

· let's define the oscillation length

$$l_{ij} = \frac{2\pi}{|E_i - E_j|} \simeq \frac{4\pi p}{|m_i^2 - m_j^2|}$$

and use $t \simeq x$

· to illustrate, consider just two neutrino generations with $\left[P_{\alpha i}\right] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$\Rightarrow P_{\nu_e \to \nu_\mu}(x) = 2\sin^2\theta\cos^2\theta - \sin^2\theta\cos^2\theta\left(e^{-i(E_1-E_2)x} + e^{-i(E_2-E_1)x}\right)$$

$$= 2\sin^2\theta\cos^2\theta\left[1 - \cos\left[(E_1 - E_2)x\right]\right]$$

$$= \sin 2\theta \cdot \sin^2\frac{\pi x}{l_{12}}$$

↪ we see that for a given mixing angle $\theta > 0$, the probability of finding a different neutrino is maximal at $x = \frac{l_{12}}{2}$. After l_{12} the neutrino will oscillate back to its original flavor. For a given momentum, the oscillation length is sensitive only to the difference of squares of neutrino masses, $l_{ij} \sim \frac{1}{|m_i^2 - m_j^2|}$

e) $\mathcal{L}^{mass} = -m\,\overline{\nu_L}\,\nu_R - \frac{M}{2}\,\overline{\nu_R^c}\,\nu_R + h.c.$ | let's assume real masses for simplicity

· let's define $N_L \equiv \nu_R^c = C\,\overline{\nu_R}^T$

$\Rightarrow \nu_R = C\,\overline{N_L}^T$

~~$\mathcal{L}^{mass}$~~ · $\overline{\nu_R}\,\nu_L = N_L^T\,C\,\nu_L$

· $\left(\overline{\nu_R^c}\,\nu_R\right)^+ = \overline{\nu_R}\,\nu_R^c = N_L^T C N_L$

$$\Rightarrow \mathcal{L}^{mass} = -m\,N_L^T C \nu_L - \frac{M}{2} N_L^T C N_L + h.c.$$

$$= -\frac{1}{2}\,n_L^T\,C\,\mathcal{M}\,n_L + h.c.$$

with $n_L = \begin{pmatrix} \nu_L \\ N_L \end{pmatrix}$ and $\mathcal{M} = \begin{pmatrix} 0 & m \\ m & M \end{pmatrix}$

(where we used $\frac{1}{2} N_L^T C \nu_L = -\frac{1}{2}\nu_L^T C^T N_L$
$= \frac{1}{2}\nu_L^T C N_L$)

· $\mathcal{M}$ can be diagonalized by $O^T \mathcal{M} O = \mathcal{M}_d$ with an orthogonal matrix O. You can check that $\mathcal{M}_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ with $m_{1,2} = \frac{M}{2} \mp \sqrt{\frac{M^2}{4} + m^2}$

and the mass eigenstates χ_{1L} and χ_{2L} are given by

$$\begin{pmatrix} \chi_{1L} \\ \chi_{2L} \end{pmatrix} = O^T n_L = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_L \\ N_L \end{pmatrix}$$

with the mixing angle θ given by

$$\tan 2\theta = \frac{2m}{M}$$

· note that χ_1 and χ_2 are Majorana particles

↪ in the limit $m \ll M$, we find

$$|m_1| \simeq \frac{m^2}{M} \ll |m_2| \simeq M$$

· now we could repeat our discussion leading to the PMNS-matrix. So the neutrino oscillations will be due not only to flavor mixing, but also to the mixing btw. ν_L and N_L. For $m \ll M$, however, the mixing angle will be very small.

· note that l_{ij} is also momentum-dependent, so for small enough neutrino momenta, l_{ij} becomes small compared to x (distance sun-earth), so one would average over the distance and get a const. probability $\langle P_{\nu_e \to \nu_\mu} \rangle$. For large momenta this is not possible. In fact, for large momenta the high electron density in the sun is crucial for the correct calculation of the neutrino oscillations to match observations, but this lies beyond the scope of this exercise.