"QCD and Standard Model"
Problem Set 9

1. Flavor parameters and CKM matrix

Let us now focus on the Yukawa interactions for leptons and quarks

$$\mathcal{L}_Y = -\Lambda_{ij}^{(e)} \bar{E}_L^i H e_R^j - \Lambda_{ij}^{(d)} \bar{Q}_L^i H d_R^j - \Lambda_{ij}^{(u)} \bar{Q}_L^i \tilde{H} u_R^j + \text{h.c.} \,, \tag{1}$$

where i, j are family indices.

a) Rotate the quark fields in order to diagonalize the Yukawa interactions. Show that the transformations on the right-handed quarks are unphysical but the ones on the left-handed quarks cannot be rotated away : in the diagonal (weak) basis they give rise to a mixing matrix (the CKM matrix) in the charged-current sector of the theory.

b) Given an arbitrary number N of quark families, determine the number of independent real parameters (mixing angles) and imaginary parameters (complex phases) of the CKM matrix. Make sure that the complex phases are really independent, *i.e.*, they cannot be reabsorbed into quark field redefinitions. Apply the formula when $N = 2$ and $N = 3$.

c) Make a CP transformation of the Yukawa term and convince yourselves that invariance under it implies that the Yukawa matrices must satisfy $\Lambda = \Lambda^*$. The existence of complex phases therefore points at CP violation. Using the results of point (b), justify why the experimental evidence of CP violation was a strong indication of the existence of a third generation of quarks.

d) Show that if neutrinos are massless, there is no mixing matrix in the lepton sector.

2. The (massive) neutrino

As we saw, neutrinos are taken to be massless in the (canonical or minimal) Standard Model. If this were actually true, the number of electron neutrinos reaching the Earth from the Sun should be twice of what it is observed. The *solar neutrino problem* can be explained if neutrinos experience flavor oscillations, a phenomenon that requires them to have non-zero masses. In other words, observations suggest that we should go beyond the Standard Model and modify it accordingly in order to accommodate neutrino masses.

The purpose of this exercise is to go through this topic to give some clarifications.

a) Discuss whether neutrinos can be Dirac or Majorana particles.

b) Use these results to generalize the Yukawa part of the electroweak Lagrangian density $\mathcal{L}_Y$ given in the previous exercise, see Eq. (1), by writing renormalizable mass terms for the neutrinos.

c) If neutrinos are massive, leptons can oscillate in flavour as the quarks do. To see it, we repeat the same steps which brought us to define the CKM matrix. Apply unitary transformations on the leptons in order to rotate them from the flavour to the mass basis. Then, show that this implies flavour mixing in the charged current sector of the electroweak theory.
What you just found is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix.

d) The intertwining between non-zero masses and flavour oscillations can be enlightened by a simple quantum mechanical argument. Expand the neutrino mass eigenstates in plane waves and compute the flavour mixing phase. Discuss the dependence of this quantity on the masses.

e) The most well-known mechanism offering an explanation of why neutrinos are so incredibly light as compared to the quarks and the other leptons, is the so-called "seesaw mechanism." Show that the Dirac and Majorana mass terms for neutrinos can be recast in the following form

$$\mathcal{L}^{\rm mass}[N] \; = \; -m N_L^T C \nu_L - \frac{M}{2} N_L^T C N_L + h.c. \; , \tag{2}$$

if one introduces the left handed spinor

$$N_L = \nu_R^c = C \bar{\nu}_R^T. \tag{3}$$

For the sake of simplicity, work with one family. Write Eq. (2) in matrix form and find the eigenvalues of the corresponding mass matrix. Estimate the value of the smallest eigenvalue, assuming that $m \lll M$. Can you explain why this is a reasonable assumption ?