QCD & SM - Problem Set 8

Problem ①

ⓐ The matrix element reads:

$$\mathcal{M} = \frac{g}{\sqrt{2}} \varepsilon_\mu \bar{e}(\vec{p}) \gamma_\mu L \nu(\vec{q}) \quad ⊛, \quad L = \frac{1}{2}(1+\gamma_5)$$

$$\rightsquigarrow \mathcal{M}^\dagger = \frac{g}{\sqrt{2}} \varepsilon_\mu^* \nu^\dagger(\vec{q}) L^\dagger \gamma_\mu^\dagger \bar{e}^\dagger(\vec{p})$$

$$= \frac{g}{\sqrt{2}} \varepsilon_\mu^* \nu^\dagger(\vec{q}) L^\dagger \gamma_\mu^\dagger (e^\dagger(\vec{p})\gamma_0)^\dagger$$

$$= \frac{g}{\sqrt{2}} \varepsilon_\mu^* \nu^\dagger(\vec{q}) L \gamma^0 \gamma_\mu e(\vec{p})$$

where we used $L^\dagger = L$, $\gamma_\mu^\dagger = \gamma^0\gamma_\mu\gamma^0$, $\gamma^{0\dagger} = \gamma^0$, $(\gamma^0)^2 = 1$

Starting from the definition of L,

we notice that

$$L\gamma^0 = \left(\frac{1+\gamma_5}{2}\right)\gamma_0 = \gamma_0\left(\frac{1-\gamma_5}{2}\right) = \gamma_0 R,$$

since the γ-matrices anticommute with γ_5.

We find

$$\mathcal{M}^\dagger = \frac{g}{\sqrt{2}} \varepsilon_\mu^* \bar{\nu}(\vec{q}) R \gamma_\mu e(\vec{p}) \quad ⊛⊛$$

Putting together ⊛, ⊛⊛, we obtain

$$|\mathcal{M}|^2 = \sum_{\text{spins}} \mathcal{M}^\dagger \mathcal{M} = \frac{g^2}{2} \varepsilon_\mu^* \varepsilon_\nu \sum_{\text{spins}} \bar{v}(\vec{q}) R \gamma_\mu e(\vec{p}) \bar{e}(\vec{p}) \gamma_\nu L v(\vec{q})$$

$$= \frac{g^2}{2} \varepsilon_\mu^* \varepsilon_\nu \, Tr(\not{q} R \gamma_\mu \not{p} \gamma_\nu L)$$

$$= \frac{g^2}{2} \varepsilon_\mu^* \varepsilon_\nu \, q_\alpha p_\beta \, Tr(\gamma_\alpha R \gamma_\mu \gamma_\beta \gamma_\nu L)$$

$$= \frac{g^2}{2} \varepsilon_\mu^* \varepsilon_\nu \, q_\alpha p_\beta \, Tr\left[\gamma_\alpha \left(\frac{1-\gamma_5}{2} \right) \gamma_\mu \gamma_\beta \gamma_\nu \left(\frac{1+\gamma_5}{2} \right) \right]$$

$$= g^2 \varepsilon_\mu^* \varepsilon_\nu \, q_\alpha p_\beta \left(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\beta} g_{\mu\nu} + g_{\alpha\nu} g_{\mu\beta} + i \varepsilon_{\alpha\mu\beta\nu} \right)$$

$$= g^2 \Big[(q \cdot \varepsilon^*)(p \cdot \varepsilon) - (\varepsilon^* \cdot \varepsilon)(p \cdot q) + (q \cdot \varepsilon)(p \cdot \varepsilon^*)$$

$$+ \underbrace{i \varepsilon_{\alpha\mu\beta\nu} q_\alpha \varepsilon_\mu^* p_\beta \varepsilon_\nu}_{\hookrightarrow \text{parity violating term!}} \Big] \qquad (***)$$

Using $\varepsilon_T^\mu(+) = \frac{1}{\sqrt{2}}(0; 1, i, 0)$ &

$p_\mu = \frac{m_W}{2}(1, \sin\vartheta, 0, \cos\vartheta)$, $q_\mu = \frac{m_W}{2}(1, -\sin\vartheta, 0, -\cos\vartheta)$,

it's straightforward to see that

$$|\mathcal{M}(+)|^2 = \frac{g^2 m_W^2}{4} (1 - \cos\theta)^2$$

From the above, we find

$$\frac{d\Gamma(+)}{d\Omega} = \frac{1}{64\pi^2 m_W} |\mathcal{M}(+)|^2$$

$$= \frac{g^2 m_W}{64\pi^2} \times \frac{1}{4}(1-\cos\theta)^2$$

Integrating over the angles, we find

$$\Gamma(+) = \int d\Omega \frac{d\Gamma(+)}{d\Omega} = \dots = \frac{g^2 m_W}{48\pi}.$$

$$\left(\int d\Omega = \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \right)$$

(b) For W polarized along the negative z axis, we start from (***) & use

$$\varepsilon_T^\mu(-) = \frac{1}{\sqrt{2}}(0; 1, -i, 0) = \varepsilon_T^{\mu *}(+),$$

thus the only difference we expect w.r.t. $|\mathcal{M}(+)|^2$ will be related to the term proportional to the Levi-Civita symbol. Actually, it's trivial to see that only its sign changes. A careful computation reveals that

$$|\mathcal{M}(-)|^2 = \frac{g^2 m_W^2}{4}(1+\cos\theta)^2 .$$

For the total decay rate we see

$$\Gamma(-) = \int d\Omega \, \frac{d\Gamma(-)}{d\Omega} = \dots = \Gamma(+),$$

as it should of course. The decay rate cannot depend on how we chose the axis.

(c) For the longitudinal polarization, we also start from (***) & this time use

$$\varepsilon_L^\mu(0) = (0;0,0,1)$$

We notice that the term proportional to the Levi-Civita symbol vanishes identically. It's easy to see that

$$|\mathcal{M}(0)|^2 = \frac{g^2 m_W^2}{2}\sin^2\theta ,$$

meaning that

$$\frac{d\Gamma(0)}{d\Omega} = \frac{g^2 m_W^2}{64\pi^2} \times \frac{1}{2}\sin^2\theta$$

$\rightsquigarrow \Gamma(0) = \Gamma(+) = \Gamma(-)$, also as it should.

(d) For an unpolarized W, we have to take the average of all possible polarizations:

$$|\mathcal{M}|^2 = \frac{1}{3}\left(|\mathcal{M}(+)|^2 + |\mathcal{M}(-)|^2 + |\mathcal{M}(0)|^2\right)$$

$$= \frac{g^2 m_W^2}{3}\left(\frac{1}{4}(1-\cos\theta)^2 + \frac{1}{4}(1+\cos\theta)^2 + \frac{1}{2}\sin^2\theta\right) = \frac{g^2 m_W^2}{3}$$

$$\rightsquigarrow \Gamma = \int d\Omega \frac{d\Gamma}{d\Omega} = \Gamma(+) = \Gamma(-) = \Gamma(0).$$

(e) For the leptonic channel, we have

$$\Gamma(W \to e\nu_e) \simeq \Gamma(W \to \mu\nu_\mu) \simeq \Gamma(W \to \tau\nu_\tau)$$

$$\rightsquigarrow \Gamma(W \to \text{leptons}) = 3\,\Gamma(W \to e\nu_e).$$

For the hadronic channel, we find

$\Gamma(W \to \bar{q}q') \simeq 6\,\Gamma(W \to e\nu_e)$, since the W-boson cannot decay to the top quark.

Therefore,

$$\Gamma(W \to \bar{s}M) \simeq 9\,\Gamma(W \to e\bar{\nu}_e)$$

$$\simeq \frac{g^2}{4\pi} \cdot \frac{3m_W}{4} \simeq 2\,\mathrm{GeV}.$$

Problem ②

The Yukawa interaction term is

$$\mathcal{L}_Y = \frac{g}{2}\frac{m_f}{M_W} h \bar{f} f = y h \bar{f} f, \text{ where}$$

to keep the expression short, we introduced

$$y = \frac{g}{2}\frac{m_f}{M_W}.$$

The (tree-level) amplitude reads

$$\mathcal{M} = y\, \bar{u}_{s_1}(\vec{p}_1) v_{s_2}(\vec{p}_2),$$

with $\left\{ \begin{matrix} \vec{p}_1, s_1 \\ \vec{p}_2, s_2 \end{matrix} \right\}$ the momenta & spin of the outgoing fermion & antifermion

Then,

$$\sum_{spin} |\mathcal{M}|^2 = y^2 \sum_{spins} \bar{u}_{s_1}(\vec{p}_1) v_{s_2}(\vec{p}_2) \bar{v}_{s_2}(\vec{p}_2) u_{s_1}(\vec{p}_1)$$

$$= y^2 \, Tr[(\not{p}_2 - m_f)(\not{p}_1 + m_f)]$$

$$= y^2 \, Tr[\not{p}_2 \not{p}_1 - m_f^2] \qquad (Tr(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu})$$

$$= 4y^2 (p_1 \cdot p_2 - m_f^2)$$

In the center-of-mass frame,

$$k^\mu = (m_h, \vec{0}), \quad p_1^\mu = \left(\frac{m_h}{2}, \vec{p}\right), \quad p_2^\mu = \left(\frac{m_h}{2}, -\vec{p}\right)$$

since from 4-momentum conservation

$$m_h = 2E_f, \quad E_f^2 = \vec{p}^{\,2} + m_f^2$$

Then

$$\sum_{spins} |\mathcal{M}|^2 = 4y^2 \left(\frac{m_h^2}{2} - 2m_f^2\right) = 2y^2 m_h^2 \left(1 - 4\frac{m_f^2}{m_h^2}\right)$$

Having computed the amplitude, the total decay rate

$$\Gamma = \frac{N_c}{8\pi m_h^2} |\vec{p}| |\mathcal{M}|^2 = N_c \frac{y^2}{8\pi} m_h \left(1 - \frac{4m_f^2}{m_h^2}\right),$$

since

$$|\vec{p}| = \frac{m_h}{2}\left(1 - \frac{4m_f^2}{m_h^2}\right)$$

Therefore,

$$\Gamma = m_h \frac{N_c\, g^2 m_f^2}{32\pi\, M_W^2}\left(1 - \frac{4m_f^2}{m_h^2}\right)^{3/2}$$