

Fermi theory: a low energy description of weak interaction

I: The charged sector of 4-Fermi theory

(a) The Lagrangian for the electroweak interactions in the Standard Model (without the Yukawa sector) is:

$$\mathcal{L}_{SM} = -\frac{1}{4} W_{\mu\nu}^a W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu H)^\dagger (D^\mu H) - \lambda \left(H^\dagger H - \frac{v^2}{2} \right)^2 + i \sum_j \bar{\psi}_j \not{D} \psi_j$$

where $W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g \epsilon^{abc} W_\mu^b W_\nu^c$,

$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$,

$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, H_{1,2} \in \mathbb{C}$,

$D_\mu = \partial_\mu - ig W_\mu^a \tau^a - ig' \frac{Y}{2} B_\mu$,

$\psi_j = \{ Q_L, E_L, u_R, d_R, e_R \}, Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L, E_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L$

That is, we have added $\mathcal{L}_f = i \sum_j \bar{\psi}_j \not{D} \psi_j$ to the $SU(2) \times U(1)$ theory from before.

(b) $\mathcal{L}_D = \bar{\psi} i \gamma_\mu \partial^\mu \psi \rightsquigarrow$ symmetries: $\psi \rightarrow \psi' = e^{i\alpha} \psi = \psi + \alpha \overset{= \delta_V \psi}{i \psi}$
 $\psi \rightarrow \psi' = e^{i\gamma_5 \beta} \psi = \psi + \beta \overset{= \delta_A \psi}{i \gamma_5 \psi}$

Noether current: $J^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \delta \varphi$ w/ φ any kind of field

$\Rightarrow \boxed{J_V^\mu = i \bar{\psi} \gamma^\mu \delta_V \psi = -\bar{\psi} \gamma^\mu \psi}$ Vector current

$\boxed{J_A^\mu = i \bar{\psi} \gamma^\mu \delta_A \psi = -\bar{\psi} \gamma^\mu \gamma_5 \psi}$ Axial current

Experiment: Wu et al '56

Parity is maximally violated in the β -decay, i.e. the weak interaction couples only to left handed fermions.

$\Rightarrow \boxed{J_L^\mu = \frac{1}{2} (J_V^\mu + J_A^\mu) = -\bar{\psi} \gamma^\mu \frac{1}{2} (1 + \gamma_5) \psi = -\bar{\psi} \gamma^\mu P_L \psi = -\bar{\psi}_L \gamma^\mu \psi_L}$ "V-A theory"

"V-A was the key", Steven Weinberg (see "Weinberg S., Essay: Half a century of the SM, Phys. rev. Lett. 2018" for more information)

Note: We work with $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ instead of $\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

In the latter case, we would have a relative minus between the currents, therefore justifying the name V-A. (Definitions in Weyl basis).

(c) $\mathcal{L}_{SM} = \bar{Q}_L i \not{D} Q_L + \bar{E}_L i \not{D} E_L$ w/ $Q_L = \begin{pmatrix} u \\ d \end{pmatrix}$, $E_L = \begin{pmatrix} \nu \\ e \end{pmatrix}$

Consider the first term (second term is analogous) in the following.

$$\bar{Q}_L i \not{D} Q_L = \bar{Q}_L i \gamma^\mu \left(\partial_\mu - ig W_\mu^\alpha \tau^\alpha - ig' \frac{Y(Q_L)}{2} B_\mu \right) Q_L$$

$$= -i \begin{pmatrix} \frac{1}{2} (g' Y(Q_L) B_\mu + g W_\mu^3) & \frac{1}{2} (g W_\mu^1 - ig W_\mu^2) \\ \frac{1}{2} (g W_\mu^1 + ig W_\mu^2) & \frac{1}{2} (g' Y(Q_L) B_\mu - g W_\mu^3) \end{pmatrix} =$$

$$W_\mu^\pm = \frac{1}{\sqrt{2}} (W_\mu^1 \mp i W_\mu^2) \rightarrow = -i \begin{pmatrix} \dots & \frac{g}{\sqrt{2}} W_\mu^+ \\ \frac{g}{\sqrt{2}} W_\mu^- & \dots \end{pmatrix}$$

$\Rightarrow \bar{Q}_L i \not{D} Q_L = \frac{g}{\sqrt{2}} W_\mu^+ (\bar{u}_L \gamma^\mu d_L) + \frac{g}{\sqrt{2}} W_\mu^- (\bar{d}_L \gamma^\mu u_L) + \text{diagonal part}$

Analogous for the leptons. In total, i.e. for the sum of the two, we find

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} (W_\mu^+ J^{\mu+} + W_\mu^- J^{\mu-}) \quad \text{w/} \quad J^{\mu+} = \bar{u}_L \gamma^\mu d_L + \bar{\nu}_L \gamma^\mu e_L,$$

$$J^{\mu-} = \bar{d}_L \gamma^\mu u_L + \bar{e}_L \gamma^\mu \nu_e$$

Since the currents are purely left-handed, they respect the prescription of (b). Note that under Hermitian conjugation $(W_\mu^+)^{\dagger} = W_\mu^-$ & $(J_\mu^+)^{\dagger} = J_\mu^-$.

Note: In the literature you may find J_μ^+ & J_μ^- defined w/ opposite signs, i.e.

$$J^{\mu+} = \bar{d}_L \gamma^\mu u_L + \bar{e}_L \gamma^\mu \nu_e$$

$$J^{\mu-} = \bar{u}_L \gamma^\mu d_L + \bar{\nu}_L \gamma^\mu e_L$$

(d) We need to express \mathcal{L} in terms of W^\pm .

Mass term From the previous problem set we know that after SSB the mass term is

$$m_W^2 W_\mu^+ W^{\mu-} \quad w/ \quad m_W = \frac{gV}{2}$$

Kinetic term • $W_\mu^\pm = \frac{1}{\sqrt{2}} (W_\mu^1 \mp iW_\mu^2)$

$$\Rightarrow W_\mu^1 = \frac{1}{\sqrt{2}} (W_\mu^+ + W_\mu^-)$$

$$W_\mu^2 = \frac{i}{\sqrt{2}} (W_\mu^+ - W_\mu^-)$$

Consider only Abelian part of field strength $W_{\mu\nu}^\alpha$ here, the non-Abelian part will play no role in the low energy theory:

$$\begin{aligned} \bullet (W_{\mu\nu}^1)^2 &= (\partial_\mu W_\nu^1 - \partial_\nu W_\mu^1)^2 = \frac{1}{2} (\partial_\mu W_\nu^+ + \partial_\mu W_\nu^- - \partial_\nu W_\mu^+ - \partial_\nu W_\mu^-)^2 \\ &= \frac{1}{2} (W_{\mu\nu}^+ + W_{\mu\nu}^-)^2 \end{aligned}$$

$$\begin{aligned} \bullet (W_{\mu\nu}^2)^2 &= (\partial_\mu W_\nu^2 - \partial_\nu W_\mu^2)^2 = -\frac{1}{2} (\partial_\mu W_\nu^+ - \partial_\mu W_\nu^- - \partial_\nu W_\mu^+ + \partial_\nu W_\mu^-)^2 \\ &= -\frac{1}{2} (W_{\mu\nu}^+ - W_{\mu\nu}^-)^2 \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathcal{L} &\supset -\frac{1}{4} W_{\mu\nu}^\alpha W^{\mu\nu\alpha} \supset -\frac{1}{4} (W_{\mu\nu}^1)^2 - \frac{1}{4} (W_{\mu\nu}^2)^2 \\ &= -\frac{1}{8} \left((W_{\mu\nu}^+)^2 + (W_{\mu\nu}^-)^2 + 2W_{\mu\nu}^+ W^{\mu\nu-} \right) + \frac{1}{8} \left((W_{\mu\nu}^+)^2 + (W_{\mu\nu}^-)^2 - 2W_{\mu\nu}^+ W^{\mu\nu-} \right) \\ &= -\frac{1}{2} W_{\mu\nu}^+ W^{\mu\nu-} \end{aligned}$$

In total, neglecting higher order interactions, we obtain:

$$\mathcal{L}[W^\pm] = -\frac{1}{2} W_{\mu\nu}^+ W^{\mu\nu-} + m_W^2 W_\mu^+ W^{\mu-} + \frac{g}{\sqrt{2}} (W_\mu^+ J^{\mu+} + W_\mu^- J^{\mu-})$$

EOM: $\partial_\mu W^{\mu\nu-} + m_W^2 W^{\nu-} = -\frac{g}{\sqrt{2}} J^{\nu+}$

& $m_W^2 \partial_\nu W^{\nu-} = -\frac{g}{\sqrt{2}} \partial_\nu J^{\nu+} \stackrel{\uparrow}{=} 0$ as constraint
conserved current

impose constraint $\Rightarrow (\square + m_W^2) W_\nu^- = -\frac{g}{\sqrt{2}} J_\nu^+$

Solution: $W^{\mu-} = -\frac{g}{\sqrt{2}} \frac{J^{\mu+}}{\square + m_W^2} \stackrel{\text{low energy}}{\simeq} -\frac{g}{\sqrt{2} m_W^2} J^{\mu+}$ & similarly for $W^{\mu+}$
 $E \ll m_W$

Insert back into \mathcal{L} :

$$\mathcal{L}_{\text{low energy}} = -\frac{g^2}{2m_W^2} J_\mu^+ J^{\mu-} = -2\sqrt{2} G_F J_\mu^+ J^{\mu-} + \mathcal{O}\left(\frac{1}{m_W^4}\right)$$

$$\text{w/ } G_F = \frac{g^2}{4\sqrt{2} m_W^2}$$

This is the Lagrangian of the famous 4-Fermi theory.
 (The numerical factors in the definition of G_F come from the comparison with the original Fermi-Lagrangian and thus are of historic origin.)

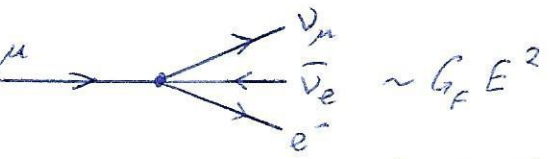
(e) $[G_F] = -2 \rightarrow$ not renormalizable

Reminder: [see e.g. Peskin-Schroeder, 10.1]

A theory w/ a coupling constant α is:

- Super-Renormalizable: $[\alpha] > 0$
- Renormalizable: $[\alpha] = 0$
- Non-Renormalizable: $[\alpha] < 0$

(f) $\mu \rightarrow e^- + \nu_\mu + \bar{\nu}_e$



$$\Rightarrow \Gamma \sim |\mathcal{M}|^2 E \sim G_F^2 E^5 \sim \left(\frac{g^2}{m_W^2}\right) E^5 \stackrel{m_W \sim gv}{\sim} \frac{E^5}{v^4}$$

The energy scale in the β -decay of a muon is given by

$$E \sim m_\mu - m_e \approx m_\mu$$

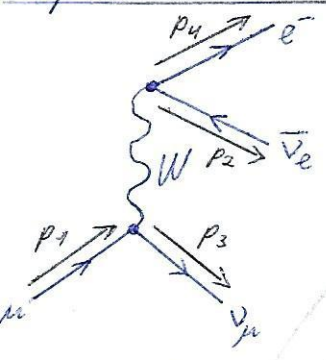
$$\Rightarrow \Gamma \sim \frac{m_\mu^5}{v^4} = \frac{1}{\tau_\mu} \Leftrightarrow \boxed{v \sim (m_\mu^5 \cdot \tau_\mu)^{1/4}}$$

Inserting $m_\mu \sim 10^2 \text{ MeV} = 10^{-1} \text{ GeV}$,
 $\tau_\mu \sim 10^{-6} \text{ s} \sim 10^{-6} \cdot \frac{10^{15} \text{ eV}}{\text{eV}} \sim \frac{10^{-18}}{\text{GeV}}$, we find

$$v \sim (10^{-5} \text{ GeV}^5 \cdot 10^{18} \text{ GeV}^{-1})^{1/4} \sim (10^{13} \text{ GeV}^4)^{1/4} \sim 10^3 \text{ GeV} \quad \underline{\text{lose enough!}}$$

[Exact values: $\Gamma \approx \frac{G_F^2 m_\mu^5}{192 \pi^3}$, $v \approx 246 \text{ GeV}$, see Griffiths - Introduction to Elementary Particles; 9.2]

Explicit calculation:



$$\begin{aligned}
 M &= 2\sqrt{2} G_F [\bar{u}_3 \gamma^\mu P_L u_1] [\bar{u}_4 \gamma_\mu P_L v_2] \\
 M^\dagger &= 2\sqrt{2} G_F [u_1^\dagger P_L^\dagger \gamma^{\mu\dagger} \gamma_0^\dagger u_3] [v_2^\dagger P_L^\dagger \gamma_\mu^\dagger \gamma_0^\dagger u_4] = \\
 &= 2\sqrt{2} G_F [\bar{u}_1 \gamma_0 P_L \gamma_0 \gamma^\mu u_3] [\bar{v}_2 \gamma_0 P_L \gamma_0 \gamma_\mu u_4] \stackrel{\{\gamma_5, \gamma^\mu\}=0}{=} \\
 &= 2\sqrt{2} G_F [\bar{u}_1 \underbrace{\gamma_0^2}_{=1} P_R \gamma^\mu u_3] [\bar{v}_2 \underbrace{\gamma_0^2}_{=1} P_R \gamma_\mu u_4] \Rightarrow P_L \gamma_\mu = \gamma_\mu P_R
 \end{aligned}$$

reminder:
 $\bar{u} = u^\dagger \gamma_0$
 $\gamma_0^\dagger = \gamma_0$
 $\gamma_0^2 = 1$
 $P_L^\dagger = P_L$
 $(\gamma^\mu)^\dagger = \gamma_0 \gamma^\mu \gamma_0$

$$\Rightarrow |M|^2 = (2\sqrt{2} G_F)^2 [\bar{u}_3 \gamma^\mu P_L u_1] [\bar{u}_1 P_R \gamma^\nu u_3] [\bar{u}_4 \gamma_\mu P_L v_2] [\bar{v}_2 P_R \gamma_\nu u_4]$$

Sum over spins, average over initial ones.
 The muon has two possible polarizations $\rightarrow \frac{1}{2}$.

$$\begin{aligned}
 \Rightarrow \langle |M|^2 \rangle &= \frac{1}{2} (2\sqrt{2} G_F)^2 \sum_{\text{all spins}} \bar{u}_3^i (\gamma^\mu P_L)_{ij} u_1^j \bar{u}_1^k (P_R \gamma^\nu)_{kl} u_3^l \cdot \\
 &\quad \bar{u}_4^a (\gamma_\mu P_L)_{ab} v_2^b \bar{v}_2^c (P_R \gamma_\nu)_{cd} u_4^d = \\
 &\quad \text{Casimir trick}
 \end{aligned}$$

$\begin{cases} P_L P_R = 0 \\ P_L \gamma_\mu = \gamma_\mu P_R \\ P_L^2 = P_L \end{cases}$

$$\begin{aligned}
 &= \frac{1}{2} \cdot 8 G_F^2 \text{tr} [P_3 \gamma^\mu P_L (\not{p}_1 - m_\mu) P_R \gamma^\nu] \text{tr} [P_4 \gamma_\mu P_L P_2 P_R \gamma_\nu] = \\
 &= 4 G_F^2 \text{tr} [P_3 \gamma^\mu P_L \not{p}_1 \gamma^\nu] \text{tr} [P_4 \gamma_\mu P_L P_2 \gamma_\nu] = \\
 &= \frac{4}{4} G_F^2 \text{tr} [P_3 \gamma^\mu (1 + \gamma_5) \not{p}_1 \gamma^\nu] \text{tr} [P_4 \gamma_\mu (1 + \gamma_5) P_2 \gamma_\nu]
 \end{aligned}$$

1st trace: $p_{3\alpha} p_{1\beta} \text{tr} [\gamma^\alpha \gamma^\mu (1 + \gamma_5) \gamma^\beta \gamma^\nu] =$

$$\begin{aligned}
 &= p_{3\alpha} p_{1\beta} [4(\gamma^{\alpha\mu} \gamma^{\beta\nu} - \gamma^{\alpha\beta} \gamma^{\mu\nu} + \gamma^{\alpha\nu} \gamma^{\beta\mu}) + 4i \epsilon^{\alpha\mu\beta\nu}] = \\
 &= 4(p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - (p_3 \cdot p_1) \gamma^{\mu\nu} + i p_{3\alpha} p_{1\beta} \epsilon^{\alpha\mu\beta\nu})
 \end{aligned}$$

2nd trace: re-label $\begin{matrix} 3 \rightarrow 4 \\ 1 \rightarrow 2 \end{matrix}$

$$= 4(p_4^\mu p_2^\nu + p_4^\nu p_2^\mu - (p_4 \cdot p_2) \gamma^{\mu\nu} + i p_{4\alpha} p_{2\beta} \epsilon^{\alpha\mu\beta\nu})$$

$$\begin{aligned}
 \Rightarrow \langle |M|^2 \rangle &= 16 G_F^2 (p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - (p_3 \cdot p_1) \gamma^{\mu\nu} + i p_{3\alpha} p_{1\beta} \epsilon^{\alpha\mu\beta\nu}) \cdot \\
 &\quad (p_{4\mu} p_{2\nu} + p_{4\nu} p_{2\mu} - (p_4 \cdot p_2) \gamma_{\mu\nu} + i p_{4\alpha} p_{2\beta} \epsilon^{\alpha\mu\beta\nu})
 \end{aligned}$$

$$\langle |M|^2 \rangle = 16 G_F^2 \left[(p_3 \cdot p_4)(p_1 \cdot p_2) + (p_3 \cdot p_2)(p_1 \cdot p_4) - (p_3 \cdot p_1)(p_4 \cdot p_2) + \right. \\ \left. + (p_3 \cdot p_2)(p_1 \cdot p_4) + (p_3 \cdot p_4)(p_1 \cdot p_2) - (p_3 \cdot p_1)(p_4 \cdot p_2) - \right. \\ \left. - (p_3 \cdot p_1)(p_4 \cdot p_2) - (p_3 \cdot p_1)(p_4 \cdot p_2) + 4(p_3 \cdot p_1)(p_4 \cdot p_2) + \right. \\ \left. + (-1) p_{3\alpha} p_{1\beta} p_{4\gamma} p_{2\delta} \underbrace{\varepsilon^{\alpha\mu\beta\nu} \varepsilon_{\mu\nu\gamma\delta}}_{\varepsilon_{\mu\nu\gamma\delta}} \right] = \\ = -2 (\eta^{\alpha\beta} \eta^{\gamma\delta} - \eta^{\alpha\delta} \eta^{\beta\gamma})$$

only $\varepsilon\varepsilon$ contraction not vanishing; other terms are symmetric in $\mu\nu$.

$$= 16 G_F^2 \left[2(p_3 \cdot p_4)(p_1 \cdot p_2) + 2(p_3 \cdot p_2)(p_1 \cdot p_4) + \right. \\ \left. + 2(p_3 \cdot p_4)(p_1 \cdot p_2) - 2(p_3 \cdot p_2)(p_1 \cdot p_4) \right] = \\ = 64 G_F^2 (p_1 \cdot p_2)(p_3 \cdot p_4)$$

Go to rest-frame of muon: $p_1 = \begin{pmatrix} m_\mu \\ 0 \end{pmatrix}$

$\rightarrow (p_1 \cdot p_2) = m_\mu E_2$

\rightarrow since $p_1 = p_2 + p_3 + p_4$, $(p_3 + p_4)^2 = p_3^2 + p_4^2 + 2p_3 \cdot p_4 = m_e^2 + 2p_3 \cdot p_4 =$
 $= (p_1 - p_2)^2 = p_1^2 + p_2^2 - 2p_1 \cdot p_2 = m_\mu^2 - 2p_1 \cdot p_2$

$\Rightarrow p_3 \cdot p_4 = \frac{m_\mu^2 - m_e^2}{2} - m_\mu E_2 \approx \frac{m_\mu}{2} (m_\mu - 2E_2)$

$\Rightarrow \langle |M|^2 \rangle = 32 G_F^2 m_\mu^2 E_2 (m_\mu - 2E_2) = 32 G_F^2 m_\mu^2 |\vec{p}_2| (m_\mu - 2|\vec{p}_2|)$

Decay rate: $1 \rightarrow 2 + \dots + n$

$$\Gamma = \frac{S}{2M} \int \langle |M|^2 \rangle (2\pi)^4 \delta^{(4)}(p_1 - p_2 - \dots - p_n) \prod_{j=2}^n dLIPS_j$$

w/ S : combinatorial factor for identical particles, i.e. for each identical particle of multiplicity n we divide by $n!$

M : mass of decaying particle

$$dLIPS_j = \frac{d^3 \vec{p}_j}{(2\pi)^3 2E_j} \quad \text{fit. of } E_2$$

Our case: $\Gamma = \frac{1}{2m_\mu} \int \langle |M|^2 \rangle (2\pi)^4 \delta^{(4)}(p_1 - p_2 - p_3 - p_4) \frac{d^3 \vec{p}_2}{(2\pi)^3 2|\vec{p}_2|} \frac{d^3 \vec{p}_3}{(2\pi)^3 2|\vec{p}_3|} \frac{d^3 \vec{p}_4}{(2\pi)^3 2|\vec{p}_4|}$

$\rightarrow \delta^{(4)}(p_1 - p_2 - p_3 - p_4) = \delta(m_\mu - |\vec{p}_2| - |\vec{p}_3| - |\vec{p}_4|) \delta^{(3)}(\vec{p}_2 + \vec{p}_3 + \vec{p}_4)$

Perform \vec{p}_3 integral w/ $\delta^{(3)}(\dots)$

$\Rightarrow \Gamma = \frac{1}{2m_\mu} \int \frac{\langle |M|^2 \rangle}{8 (2\pi)^5} \delta(m_\mu - |\vec{p}_2| - |\vec{p}_2 + \vec{p}_4| - |\vec{p}_4|) \frac{d^3 \vec{p}_2 d^3 \vec{p}_4}{|\vec{p}_2| |\vec{p}_2 + \vec{p}_4| |\vec{p}_4|}$

→ Do \vec{p}_2 integral. Go to spherical coordinates in mom. space w/ polar axis along \vec{p}_4 . We have

$$d^3 p_2 = |\vec{p}_2|^2 d|\vec{p}_2| \sin \theta d\theta d\varphi$$

$$|\vec{p}_2 + \vec{p}_4|^2 = |\vec{p}_2|^2 + |\vec{p}_4|^2 + 2|\vec{p}_2||\vec{p}_4| \cos \theta \equiv u^2$$

$$\Gamma = \frac{1}{2m_\mu} \int \frac{\langle |M|^2 \rangle}{8(2\pi)^5} |\vec{p}_2|^2 \frac{d|\vec{p}_2| d^3 \vec{p}_4}{|\vec{p}_2||\vec{p}_4|} \underbrace{\int_0^{2\pi} d\varphi}_{=2\pi} \underbrace{\int_0^\pi d\theta \sin \theta \frac{\delta(m_\mu - |\vec{p}_2| - |\vec{p}_4| - |\vec{p}_2|)}{u(\theta)}}_{\text{go to } u \text{ integration}}$$

$$u^2 = |\vec{p}_2|^2 + |\vec{p}_4|^2 + 2|\vec{p}_2||\vec{p}_4| \cos \theta \quad |d$$

$$\Rightarrow 2u du = -2|\vec{p}_2||\vec{p}_4| \sin \theta d\theta \quad (\Leftrightarrow) \quad \sin \theta d\theta = -\frac{u du}{|\vec{p}_2||\vec{p}_4|}$$

$$\Rightarrow \Gamma = \int \frac{\langle |M|^2 \rangle}{16 m_\mu (2\pi)^4} \frac{d^3 \vec{p}_4 d|\vec{p}_2|}{|\vec{p}_4|^2} \int_{u_-}^{u_+} du \delta(m_\mu - |\vec{p}_2| - |\vec{p}_4| - u)$$

$$w/ \quad u_{\pm} \equiv \sqrt{|\vec{p}_2|^2 + |\vec{p}_4|^2 \pm 2|\vec{p}_2||\vec{p}_4|} = |\vec{p}_2| \pm |\vec{p}_4|$$

(We used the - to change integration domain, i.e. $-\int_{u_+}^{u_-} = \int_{u_-}^{u_+}$)

For du integral, we must have $u_- < m_\mu - |\vec{p}_2| - |\vec{p}_4| < u_+$

$$\Leftrightarrow \left\{ \begin{array}{l} |\vec{p}_2| < \frac{m_\mu}{2} \\ |\vec{p}_4| < \frac{m_\mu}{2} \\ |\vec{p}_2| + |\vec{p}_4| > \frac{m_\mu}{2} \end{array} \right. : \text{lower bound} \quad \left. \begin{array}{l} \text{limits } |\vec{p}_2| \text{ \& } |\vec{p}_4| \text{ integration:} \\ |\vec{p}_2| \text{ runs from } \frac{m_\mu}{2} - |\vec{p}_4| \text{ to } \frac{m_\mu}{2} \\ |\vec{p}_4| \text{ runs from } 0 \text{ to } \frac{m_\mu}{2} \end{array} \right\}$$

$$\Rightarrow \Gamma = \int \frac{32 G_F^2 m_\mu^2}{16 m_\mu (2\pi)^4} \frac{d^3 \vec{p}_4}{|\vec{p}_4|^2} \int_{\frac{m_\mu}{2} - |\vec{p}_4|}^{\frac{m_\mu}{2}} d|\vec{p}_2| |\vec{p}_2| (m_\mu - 2|\vec{p}_2|) =$$

$$= \frac{2 G_F^2 m_\mu}{(2\pi)^4} \int \frac{d^3 \vec{p}_4}{|\vec{p}_4|^2} \left[|\vec{p}_2|^2 \left(\frac{m_\mu}{2} - \frac{2}{3} |\vec{p}_2| \right) \right]_{\frac{m_\mu}{2} - |\vec{p}_4|}^{\frac{m_\mu}{2}} =$$

$$= \frac{2 G_F^2 m_\mu}{(2\pi)^4} \int \frac{d^3 \vec{p}_4}{|\vec{p}_4|^2} |\vec{p}_4|^2 \left(\frac{m_\mu}{2} - \frac{2}{3} |\vec{p}_4| \right) = \frac{G_F^2 m_\mu^2}{(2\pi)^4} \int d^3 \vec{p}_4 \left(1 - \frac{4|\vec{p}_4|}{3 m_\mu} \right)$$

Finally, write $d^3 \vec{p}_4 = 4\pi |\vec{p}_4|^2 d|\vec{p}_4| = 4\pi E^2 dE$, w/ $E \hat{=} \text{energy of the } e^-$

$$\Rightarrow \frac{d\Gamma}{dE} = \frac{G_F^2 m_\mu^2 E^2}{4\pi^3} \left(1 - \frac{4E}{3 m_\mu} \right)$$

Energy distribution of emitted electron! (nicely matches experiments)

Total decay rate:

$$\Gamma = \frac{G_F^2 m_\mu^2}{4\pi^3} \int_0^{\frac{m_\mu}{2}} dE E^2 \left(1 - \frac{4E}{3m_\mu}\right)$$

$$= \left[\frac{E^3}{3} \left(1 - \frac{E}{m_\mu}\right) \right]_0^{\frac{m_\mu}{2}} = \frac{m_\mu^3}{3 \cdot 8} \left(1 - \frac{1}{2}\right) = \frac{m_\mu^3}{3 \cdot 16}$$

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

or, equivalently, $\tau_\mu = \frac{1}{\Gamma} = \frac{192\pi^3}{G_F^2 m_\mu^5} \stackrel{\uparrow}{=} \frac{384\pi^3 v^4}{m_\mu^5}$

$$G_F = \frac{g^2}{4\sqrt{2}m_W^2} \stackrel{\uparrow}{=} \frac{1}{\sqrt{2}v^2}$$

$$m_W = \frac{gv}{2}$$

Therefore, $v = \left(\frac{m_\mu^5 \tau_\mu}{384\pi^3} \right)^{1/4} =$

$$\approx \left(\frac{(105.7 \cdot 10^{-3} \text{ GeV})^5 \cdot 2.197 \cdot 10^{-6} \cdot \frac{10^{25}}{6.582} \text{ GeV}^{-1}}{384\pi^3} \right)^{1/4} \approx$$

$$\approx \underline{\underline{246.6 \text{ GeV}}} \quad \text{very close!}$$

In fact, $v = 246,2196(2) \text{ GeV}$.

Note: since all masses in the SM are $\sim v \times \text{coupling constant}$, the requirement of weak coupling gives an upper bound for all masses!