Problem 1

The starting point is:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi_i \partial^\mu \varphi_i \quad - V(\phi) \qquad , \quad (1)$$

with φ_i's are 3 real scalars $(i=1,2,3)$, and the potential is

$$V(\phi) = -\frac{\mu^2}{2}\phi_i\varphi_i + \frac{\lambda}{4}(\varphi_i\varphi_i)^4 \, , \quad (2)$$

always summation over repeated indexes is assumed.

(a) Introduce

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \qquad , (3)$$

in terms of which (1), becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^T \partial^\mu \Phi + \frac{\mu^2}{2}\Phi^T\Phi - \frac{\lambda}{4}(\Phi^T\Phi)^2 . \quad (4)$$

Now, for Φ, rotations act as

$$\Phi \to \Phi' = O\,\Phi \qquad , \qquad (5)$$

with O an orthogonal 3×3 matrix:

$$O^T O = \mathbb{1}. \qquad (6)$$

[From (5), we notice that

$$\Phi^T \to \Phi'^T = (O\Phi)^T = \Phi^T O^T \qquad (7)$$]

Plug (5) & (7) into (4), and see what happens:

$$\mathcal{L} \to \mathcal{L}' = \frac{1}{2}\partial_\mu \Phi'^T \partial^\mu \Phi' + \frac{\mu^2}{2}\Phi'^T\Phi' - \frac{\lambda}{4}\left(\Phi'^T\Phi'\right)^2$$

$$= \frac{1}{2}\partial_\mu \Phi^T \overset{(6)}{\cancel{O^T O}}{}^{\mathbb{1}} \partial^\mu \Phi + \frac{\mu^2}{2}\Phi^T \overset{(6)}{\cancel{O^T O}}{}^{\mathbb{1}} \Phi - \frac{\lambda}{4}\left(\Phi^T \overset{(6)}{\cancel{O^T O}}{}^{\mathbb{1}} \Phi\right)^2$$

$$= \frac{1}{2}\partial_\mu \Phi^T \partial^\mu \Phi + \frac{\mu^2}{2}\Phi^T\Phi - \frac{\lambda}{4}\left(\Phi^T\Phi\right)^2 = \mathcal{L} \qquad (8)$$

Indeed, we have invariance under 3D rotations in the internal space of fields φ_i.

[Equivalently, you may work with each component separately:

$\varphi_i \to \varphi_i' = \epsilon_{ijk} \ldots$]

(b) We have the potential

$$V(\varphi) = -\frac{\mu^2}{2}\varphi_i\varphi_i + \frac{\lambda}{4}(\varphi_i\varphi_i)^2, \quad (9)$$

and we want to minimize it.

* first we look for extrema:

$$\frac{\partial V}{\partial \varphi_1} = \frac{\partial V}{\partial \varphi_2} = \frac{\partial V}{\partial \varphi_3} = 0 \quad (10)$$

$$\left[\frac{\partial V}{\partial \varphi_i} = 0, \ \forall i\right]$$

From (10), we find two options:

either (i) $\varphi_i = 0$, or (ii) $\varphi_i\varphi_i = \frac{\mu^2}{\lambda}$ (11)

⊗ Once we have the extrema, we study whether they are minima or maxima, i.e.

$$\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j} \underset{\text{max.}}{\overset{\text{min}}{\lesseqgtr}} 0 \qquad (12)$$

For the potential (2) [(9)], we find

$$\text{(i) } \varphi_i = 0 \rightarrow \text{maximum} \qquad (13a)$$

$$\text{(ii) } \varphi_i \varphi_i = \frac{\mu^2}{\lambda} \rightarrow \text{minimum} \qquad (13b) \qquad (13)$$

I write (13b) explicitly:

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = \frac{\mu^2}{\lambda} \qquad (14)$$

2d sphere → vacuum manifold.

(i) I may take wlog

$$\varphi_1 = 0$$
$$\varphi_2 = 0 \qquad (15)$$
$$\varphi_3 \neq 0 \rightarrow \phi_3 = \frac{\mu}{\sqrt{\lambda}}$$

[$\varphi_1 \neq 0$, $\varphi_2 \neq 0$, $\varphi_3 \neq 0$ } I always have to satisfy (14).]

$SO(3) \rightarrow SO(2)$

We can show that explicitly: Take

$$\phi_1 = 0 + \theta_1 \,,\ \phi_2 = 0 + \theta_2 \,,\ \phi_3 = \frac{\mu}{\sqrt{\lambda}} + \chi \quad (16)$$

& plug into the Lagrangian, to find that the quadratic pieces read:

$$\mathcal{L}_{(2)} = \frac{1}{2}(\partial_\mu \theta_a)^2 + \frac{1}{2}(\partial_\mu \chi)^2 - \mu^2 \chi^2 \quad (17)$$

$$+ \ldots$$

$a = 1, 2$

higher-order in fields

Notice two things:

(1) $SO(2)$ invariance because θ_a's appear in a symmetry-preserving form

(2) χ is a massive field with mass
$m_\chi = \sqrt{2}\,\mu$

θ_1, θ_2 which are massless

we have a situation in which
$SO(3) \to SO(2)$

generators of $SO(n) = \frac{n(n-1)}{2}$

$SO(3) \Rightarrow 3$ generators

$SO(2) \Rightarrow 1$ generator

$\#^{\text{generators}}_{SO(3)} - \#^{\text{generators}}_{SO(2)} = 2 \rightarrow$ # massless fields

"Nambu – Goldstone bosons"

d) Gauging of $SO(3)$:

$$\left[\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}\right] \quad \Phi \to \Phi' = O\,\Phi \qquad (18)$$

$$O \to O = O(x)$$

Look at the kinetic part:

$$\partial_\mu \Phi'^T \partial^\mu \Phi' \to \partial_\mu (\Phi^T O^T) \partial^\mu (O\Phi)$$

$$\supset \Phi^T (\partial_\mu O^T) O \partial_\mu \Phi + \ldots$$

What we have to do is modify the derivative accordingly, such that it transform covariantly

$$\partial_\mu \to D_\mu = \partial_\mu + A_\mu \qquad (19)$$

gauge field [3×3 matrix]

that transforms in a manner that ensures

$$(D_\mu \Phi)' = O\, D_\mu \Phi \qquad (20)$$

(e) The masses of the gauge fields come from the covariant derivative of Φ. On top of the vacuum, we find

$$m_{A_1} = m_{A_2} = g\frac{\mu}{\sqrt{\lambda}} \quad , \quad m_{A_3} = 0 \quad . \tag{21}$$

Problem 2

(a) $$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \varphi_a \partial^\mu \varphi_a + \frac{\mu^2}{2}\varphi_a\varphi_a - \frac{\lambda}{4}(\varphi_a\varphi_a)^2, \quad (1)$$

with φ_a's 2 real scalars $[a=1,2]$.

* symmetry group $= SO(2)$ $[U(1)]$

first let's understand what happens in terms of φ_1, φ_2:

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \to \begin{pmatrix}\varphi_1'\\ \varphi_2'\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \quad (2)$$

$\Rightarrow$ $$\begin{aligned}\varphi_1' &= \cos\theta\,\varphi_1 + \sin\theta\,\varphi_2\\ \varphi_2' &= -\sin\theta\,\varphi_1 + \cos\theta\,\varphi_2\end{aligned} \quad (3)$$

* ground states (vacua):

first we find extrema

$$\frac{\partial V}{\partial \varphi_1} = \frac{\partial V}{\partial \varphi_2} = 0, \quad (4)$$

and then we check whether they correspond to minima

$$\frac{\partial^2 V}{\partial \varphi_1^2}, \frac{\partial^2 V}{\partial \varphi_2^2}, \frac{\partial^2 V}{\partial \varphi_1 \partial \varphi_2} \geq 0. \quad (5)$$

from (4) & (5) we find

$$\{\underbrace{\varphi_1 = \varphi_2 = 0}_{\text{maximum}}\} \quad \text{or} \quad \{\underbrace{\varphi_1^2 + \varphi_2^2 = \frac{\mu^2}{\lambda}}_{\text{min}}\} \quad (6)$$

As before, we have the liberty to take either

$$\varphi_1 = 0 \quad \& \quad \varphi_2 = \frac{\mu}{\sqrt{\lambda}} \qquad (7)$$

or vice versa, or whatever combination we like.

Initially we have $SO(2)$: 1 generator

$\to$ "nothing"

1 massless mode in the theory

$\nearrow$

exactly the same as in Problem 1

- Noether current associated with $SO(2)$:

$$j_\mu^0 = \frac{\delta \mathcal{L}_0}{\delta \partial_\mu \varphi_1} \delta\varphi_1 + \frac{\delta \mathcal{L}_0}{\delta \partial_\mu \varphi_2} \delta\varphi_2 \qquad (8)$$

$\delta\varphi_1, \delta\varphi_2$ are the infinitesimal $SO(2)$ rotations of the fields. From (3), we get

$$\delta\varphi_1 = \theta\varphi_2 \quad , \quad \delta\varphi_2 = -\theta\varphi_1 \quad . \qquad (9)$$

So, from (1), (8), (9), we get

$$j_\mu = \varphi_2 \partial_\mu \varphi_1 - \varphi_1 \partial_\mu \varphi_2 \quad . \qquad (10)$$

(b) We now take the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi_a \partial^\mu \varphi_a + \frac{\mu^2}{2}\varphi_a\varphi_a - \frac{\lambda}{4}(\varphi_a\varphi_a)^2 + \epsilon\, U(\varphi_1) \qquad (11)$$

$U(\varphi_1)$ = function <u>of φ_1 only</u> , (12)

and ϵ is a small parameter.

We look for minima of the potential.

$$\frac{\partial V}{\partial \varphi_1} = \frac{\partial V}{\partial \varphi_2} = 0 \quad . \qquad (13)$$

we find

$$\varphi_1(\mu^2 - \lambda\varphi_a\varphi_a) + \epsilon U' = 0 \quad \& \quad \varphi_2(\mu^2 - \lambda\varphi_a\varphi_a) = 0 \, . \qquad (14)$$

from the above we notice the following options:

(i) $\phi_2 = 0 \rightarrow \phi_1 \simeq \frac{\mu}{\sqrt{\lambda}} + \epsilon \frac{v'}{2\mu^2}$, (15)

(ii) $\phi_2 \neq 0 \rightarrow \phi_1 = 0, \; v' = 0$. (16)

In either case (i), (ii), the spectrum of the theory contains a particle whose mass is proportional to ϵ, such that when the explicit breaking vanishes it becomes the genuine NG boson of the broken generator.