1)

1) $u_{L,R}(x) \xrightarrow{L} u'_{L,R}(x) = S_{L,R}\, u_{L,R}(\Lambda^{-1}x)$

(or $u'_{L,R}(x') = S_{L,R}\, u_{L,R}(x)$, with $x' \equiv \Lambda x$)

↳ suppress spacetime dependence from now on

$S_{L,R} = e^{-i\frac{\sigma_j}{2}(\theta_j \mp i\phi_j)} \in SL(2,\mathbb{C})$ (n×n-complex matrices with determinant 1, <u>not</u> necessarily unitary)

a)

$$S^{+}_{L,R} = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\left(-\frac{i}{2}\sigma_j(\theta_j \mp i\phi_j)\right)^n\right]^{+}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\underbrace{\left[\left(-\frac{i}{2}\sigma_j(\theta_j \mp i\phi_j)\right)^{+}\right]}_{=+\frac{i}{2}\sigma_j(\theta_j \pm i\phi_j)}^{n} = e^{+\frac{i}{2}\sigma_j(\theta_j \pm i\phi_j)}$$

· $S^{+}_{L,R} S_{R,L} = e^{+\frac{i}{2}\sigma_j(\theta_j \pm i\phi_j)} \cdot e^{-i\frac{\sigma_j}{2}(\theta_j \pm i\phi_j)} = \underline{1\!\!1}$

$\Rightarrow S^{+}_{L,R} = S^{-1}_{R,L}$

· $\sigma_2 \; \sigma_j \;\; 2 \quad = \sigma_2 \left(\delta_{j2} + i\,\varepsilon_{j2k}\,\sigma_k\right)$

$= \left[\sigma_2 \delta_{2j} + i\,\underbrace{\varepsilon_{j2k}\left(\delta_{2k} + i\,\varepsilon_{2kl}\,\sigma_l\right)}_{= -(\delta_{jl} - \delta_{2j}\delta_{2l})\sigma_l \; = -\sigma_j + \sigma_2 \delta_{2j}}\right]$

$= \underbrace{\left(2\,\delta_{2j}\,\sigma_2 - \sigma_j\right)}_{= -\sigma_j^*}$, since $\sigma_1^* = \sigma_1$, $\sigma_3^* = \sigma_3$ but $\sigma_2^* = -\sigma_2$

$= -\sigma_j^*$ $\quad (*)$

$\longrightarrow \sigma_2 S_{L,R} \sigma_2 = \sum_{n=0}^{\infty} \frac{1}{n!} \, \sigma_2 \left[-\frac{i\sigma_j}{2}(\theta_j \mp i\phi_j)\right]^n \sigma_2 =$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \Big[-\frac{i}{2} \underbrace{\sigma_2 \sigma_j \sigma_2}_{\overset{(*)}{=} -\sigma_j^*} (\theta_j \mp i\phi_j) \Big]^n = e^{+\frac{i}{2}\sigma_j^* (\theta_j \pm i\phi_j)^*}$$

$$= S^*_{R,L}$$

- $$S^T_{L,R} = \left(S^+_{L,R}\right)^* \overset{(4a)}{=} \left(S^{-1}_{R,L}\right)^* = \left(S^*_{R,L}\right)^{-1}$$

$$\overset{(4b)}{=} \left(\sigma_2 S_{L,R} \sigma_2\right)^{-1} = \sigma_2^{-1} S^{-1}_{L,R} \sigma_2^{-1} = \sigma_2 S^{-1}_{L,R} \sigma_2$$

b)

- $$u_L \xrightarrow{L} S_L u_L \quad \Rightarrow u_L^* \xrightarrow{L} S_L^* u_L^* = \sigma_2 S_R \sigma_2 u_L^*$$

$$\Rightarrow (\sigma_2 u_L^*) \xrightarrow{L} S_R(\sigma_2 u_L^*) \qquad \Rightarrow (\sigma_2 u_L^*) \in (0, \tfrac{1}{2})$$

- $$v_L \xrightarrow{L} S_L v_L \quad \Rightarrow v_L^T \xrightarrow{L} v_L^T S_L^T = v_L^T \sigma_2 S_L^{-1} \sigma_2$$

$$\Rightarrow (v_L^T \sigma_2 u_L) \xrightarrow{L} v_L^T \sigma_2 S_L^{-1} \underbrace{\sigma_2 \sigma_2}_{=\mathbb{1}} S_L u_L = v_L^T \sigma_2 u_L$$

$$\Rightarrow (v_L^T \sigma_2 u_L) \in (0,0)$$

- $$u_L^+ \xrightarrow{L} u_L^+ S_L^+ = u_L^+ S_R^{-1}$$

$$\Rightarrow (u_L^+ \sigma_-^\mu u_L) \xrightarrow{L} u_L^+ S_R^{-1} \sigma_-^\mu S_L u_L$$

3)

↳ now, what is $S_R^{-1}\sigma_-^\mu S_L$?

↳ consider infinitesimal version:

$$\left[\mathbb{1}+\frac{i}{2}\sigma_j(\theta_j+i\phi_j)\right]\sigma_-^\mu\left[\mathbb{1}-\frac{i}{2}\sigma_k(\theta_k\mp i\phi_k)\right]+\mathcal{O}[\theta^2,\phi^2]$$

$$=\sigma_-^\mu-\frac{i}{2}\theta_j[\sigma_-^\mu,\sigma_j]-\frac{1}{2}\phi_j\{\sigma_-^\mu,\sigma_j\}+\mathcal{O}(\theta^2,\phi^2)$$

- $\mu=0$: $S_R^{-1}\sigma_-^0 S_L$

$$=\sigma_-^0-\phi_j\sigma_j+\dots$$

$$=\sigma_-^0-\phi_j\sigma_-^j+\dots$$

- $\mu=i$: $S_R^{-1}\sigma_-^i S_L=\sigma_-^i+\frac{i}{2}\theta_j\underbrace{[\sigma^i,\sigma_j]}_{=-2i\varepsilon_{ijk}\sigma_k}+\frac{1}{2}\phi_j\underbrace{\{\sigma^i,\sigma_j\}}_{=-2\delta_{ij}}+\dots$

$$=\sigma_-^i+\varepsilon_{ijk}\theta_j\sigma_k-\phi_i\mathbb{1}+\dots=\sigma_-^i+\varepsilon_{ijk}\theta_j\sigma_-^k-\phi_i\sigma_-^0+\dots$$

↳ now compare this to an infinitesimal Lorentz trafo of a 4-vector v^μ:

$$v^\mu\xrightarrow{L}v'^\mu=\Lambda^\mu{}_\nu v^\nu=(\delta^\mu{}_\nu+\underbrace{\omega^\mu{}_\nu}_{\text{anti-symm.}})v^\nu+\mathcal{O}(\omega^2)$$

$\Rightarrow v^0\longrightarrow v^0+\omega_{0i}v^i$

$v^i\longrightarrow v^i-\omega_{ij}v^j$
$+\omega_{0i}v^0$

(3/1) · note that for $w_{0i} > 0$ and ~~w_{ij}~~

$w_{12} > 0$, $w_{23} > 0$, $w_{31} > 0$, we have a Lorentz trafo in "positive" direction, while for $w_{0i} < 0$, $w_{12}, w_{23}, w_{31} < 0$ it is in the "negative" direction

↳ let's define $w_{0i} \equiv -\phi_i$ (with $\phi^i = -\phi_i > 0$)

and $w_{ij} \equiv \varepsilon_{ijk}\theta_k$ (with $\varepsilon^{123} = +1$ and
$= \varepsilon^{ijk}\theta^k$ $\quad \theta^k > 0$).

↳ then we see that $S_R^{-1} \sigma^\mu S_L = \Lambda^\mu{}_\nu \sigma^\nu$
(after compounding the infinitesimal trafo's to get a finite trafo)

↳ "positive" direction

$\Rightarrow (u_L^\dagger \sigma_-^\mu u_L) \xrightarrow{L} \Lambda^\mu{}_\nu (u_L^\dagger \sigma_-^\nu u_L)$

$\hookrightarrow$ so $(u_L^\dagger \sigma_-^\mu u_L)$ transforms as a vector, hence lives in the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation.

$\hookrightarrow$ similarly, $(u_R^\dagger \sigma_+^\mu u_R) \in \left(\frac{1}{2}, \frac{1}{2}\right)$.

- $\mathcal{L} = i u_L^\dagger \sigma_-^\mu \partial_\mu u_L + i u_R^\dagger \sigma_+^\mu \partial_\mu u_R + u_L^T \sigma_2 u_L + \text{h.c.}$

c)

- $\Sigma^{0i} = \frac{1}{2i} \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$ (boost generators)

- $\Sigma^{ij} = -\frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$ (rotation generators)

- $\psi = \begin{pmatrix} u_L \\ u_R \end{pmatrix} \longrightarrow S\psi = e^{i\theta_{\mu\nu}\Sigma^{\mu\nu}} \psi$

$= e^{i[2\theta_{0i}\Sigma^{0i} + \theta_{ij}\Sigma^{ij}]} \begin{pmatrix} u_L \\ u_R \end{pmatrix}$

$=$

4)

$$= \exp\left[\theta_{0i}\begin{pmatrix}-\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix} - \frac{1}{2} i\theta_{ij}\varepsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}\right]\begin{pmatrix}u_L\\ u_R\end{pmatrix}$$

since exp. is diag. matrix

$$= \begin{pmatrix} e^{-\theta_{0i}\sigma^i - \frac{1}{2}i\theta_{ij}\varepsilon^{ijk}\sigma^k} & 0\\ 0 & e^{\theta_{0i}\sigma^i - \frac{1}{2}i\theta_{ij}\varepsilon^{ijk}\sigma^k}\end{pmatrix}\begin{pmatrix}u_L\\ u_R\end{pmatrix}$$

c) if we now define

$\theta_{0i} = -\frac{\phi_i}{2}$ and $\theta_{ij}\varepsilon^{ijk} = +\theta^k$

we find

$$\begin{pmatrix}u_L\\ u_R\end{pmatrix} \to \begin{pmatrix}S_L u_L\\ S_R u_R\end{pmatrix}$$, as it should.

d) $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$

$$= \begin{pmatrix}0 & \sigma^\mu_+\\ \sigma^\mu_- & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\nu_+\\ \sigma^\nu_- & 0\end{pmatrix} + \begin{pmatrix}0 & \sigma^\nu_+\\ \sigma^\nu_- & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu_+\\ \sigma^\mu_- & 0\end{pmatrix}$$

$$= \begin{pmatrix}\sigma^\mu_+\sigma^\nu_- & 0\\ 0 & \sigma^\mu_-\sigma^\nu_+\end{pmatrix} + \begin{pmatrix}\sigma^\nu_+\sigma^\mu_- & 0\\ 0 & \sigma^\nu_-\sigma^\mu_+\end{pmatrix} = 2g^{\mu\nu}$$

$\cdot \{\gamma^\mu, \gamma_5\} = \begin{pmatrix} 0 & \sigma^\mu_+ \\ \sigma^\mu_- & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} + \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu_+ \\ \sigma^\mu_- & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & -\sigma^\mu_+ \\ \sigma^\mu_- & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^\mu_+ \\ -\sigma^\mu_- & 0 \end{pmatrix} = 0$

$\cdot \gamma^{\mu\dagger} = \begin{cases} \gamma^0 & , \mu = 0 \\ -\gamma^i & , \mu = i \end{cases}$

$\hookrightarrow$ since γ^0 commutes with γ^0, but anti-commutes with γ^i: $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$

$\gamma_5^\dagger = +i\, \gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} = -i\, \gamma^3\gamma^2\gamma^1\gamma^0$

$= -i\,(-1)^{3+2+1} \cdot \gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5$

e) $\cdot\, \psi \xrightarrow{L} S\psi \quad , \; S = e^{i\theta_{\mu\nu}\Sigma^{\mu\nu}}$

$\cdot\, \psi^\dagger \xrightarrow{L} \psi^\dagger S^\dagger \quad \Rightarrow \bar\psi \xrightarrow{L} \psi^\dagger S^\dagger \gamma^0 = \bar\psi \gamma^0 S^\dagger \gamma^0$

$\hookrightarrow \Sigma^{\mu\nu\dagger} = -\frac{1}{4i}\left[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}\right] = \frac{1}{4i}\left[\gamma^{\mu\dagger}, \gamma^{\nu\dagger}\right] \overset{d)}{=} \gamma^0 \Sigma^{\mu\nu} \gamma^0$

(5)

$$\Rightarrow \gamma^0 S^\dagger \gamma^0 = \gamma^0 e^{-i\theta_{\mu\nu}\Sigma^{\mu\nu\dagger}} \gamma^0 = e^{-i\theta_{\mu\nu}\Sigma^{\mu\nu}}$$

$$= S^{-1}$$

$$\Rightarrow \bar{\psi} \xrightarrow{L} \bar{\psi} S^{-1}$$

$$\Rightarrow \bar{\psi}\psi \to \bar{\psi} S^{-1} S \psi = \bar{\psi}\psi \quad \text{(scalar)}$$

- $\bar{\psi}\gamma^\mu\psi \xrightarrow{L} \bar{\psi} S^{-1} \gamma^\mu S \psi$

↳ following analogous steps as before, we can show that $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$

$$\Rightarrow \bar{\psi}\gamma^\mu\psi \xrightarrow{L} \Lambda^\mu{}_\nu \bar{\psi}\gamma^\nu\psi \quad \text{(vector)}$$

f) $\cdot\ L\psi = \frac{1}{2}\left[\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} + \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\right] \cdot \begin{pmatrix} u_L \\ u_R \end{pmatrix} = \begin{pmatrix} u_L \\ 0 \end{pmatrix}$

$\cdot\ R\psi = \frac{1}{2}\left[\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} - \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\right] \begin{pmatrix} u_L \\ u_R \end{pmatrix} = \begin{pmatrix} 0 \\ u_R \end{pmatrix}$

↳ if we choose a basis, where γ_5 is not diagonal, ($L/R\ \psi$) ~~...~~ will still contain upper and lower components.