$\boxed{1}$ The SU(N) Group

$\lambda^a \ll 1$

1) $1 = U^\dagger U = e^{-i\lambda^a T^{a\dagger}} e^{i\lambda^a T^a} = (1 - i\lambda^a T^{a\dagger})(1 + i\lambda^a T^a)$

$= 1 + i\lambda^a (T^a - T^{a\dagger}) + \mathcal{O}(\lambda^{a2})$

$\Longrightarrow \boxed{T^a = T^{a\dagger}}$ Hermitian

Notes: ▷ In mathematical literature exp. map is often defined w/out the i. This rescaling of the generators would lead to anti-Hermitian generators. In physics we prefere Hermitian matrices since their eigenvalues are purely real (in contrast to purely imaginary), and we like to identify hermitian matrices w/ observables.

▷ For connected liegroups ~~and compact groups the exp map is bijective so~~ every element is smoothly connected to the identity. Thus it is sufficient to look at elements close to the identity ($\Leftrightarrow \lambda^a \ll 1$). But same follows for whole group if one applies Baker-Campbell-Hausdorff formula.

$1 = \det U = \det(e^{i\lambda^a T^a}) = e^{i\lambda^a tr(T^a)}$

$\det(e^A) = e^{trA}$ for some matrix A

$\Longrightarrow \boxed{tr(T^a) = 0} \; \forall a$ traceless

So $\{T^a\}_a$ span Lie-algebra su(N) and $k = \dim(su(N))$ = # free parameters of a general su(N) element.

Counting: · Complex N×N matrix : $2N^2$

· hermitian: ▷ diagonal part pure real $\quad -N$

▷ non-diagonal part defined by upper right triangular matrix $\quad -\sum_{i=1}^{N-1} 2i \overset{Gauß}{=} -\frac{2N(N-1)}{2}$

$= -N(N-1)$

$= -N^2 + N$

▷ traceless $\quad -1$

$2N^2 - N - N^2 + N - 1 = \boxed{N^2 - 1 = k}$

2) $\boxed{su(2):}$ general element $\begin{pmatrix} a & b-ic \\ b+ic & -a \end{pmatrix}$

$\rightsquigarrow$ 3 free parameters (consistent w/ N^2-1 of course)

<u>Strategy:</u> Pick one free parameter $\neq 0$ and the others $=0$

$$T^1 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & -ic \\ +ic & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

<u>Normalize:</u> $Tr(T^1T^1) = Tr\begin{pmatrix} b^2 & 0 \\ 0 & b^2 \end{pmatrix} = 2b^2 \overset{!}{=} \frac{1}{2}$

$$\Longrightarrow \boxed{b = \pm\frac{1}{2}}$$

$Tr(T^2T^2) = Tr\begin{pmatrix} c^2 & \\ & c^2 \end{pmatrix} = 2c^2 \overset{!}{=} \frac{1}{2}$

$$\Longrightarrow \boxed{c = \pm\frac{1}{2}}$$

$Tr(T^3T^3) = Tr\begin{pmatrix} a^2 & \\ & a^2 \end{pmatrix} = 2a^2 \overset{!}{=} \frac{1}{2}$

$$\Longrightarrow \boxed{a = \pm\frac{1}{2}}$$

Pick positive for each

$$\Longrightarrow T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Longleftrightarrow \boxed{T^a = \frac{\sigma^a}{2}} \text{ w/ } \sigma^a \text{... Pauli-matrices!}$$

<u>Orthogonality:</u> Per construction of setting one free parameter $\neq 0$ and rest $=0$

[Numerating was chosen so that standard numerating of the σ's comes out]

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$su(3)$ / general element $\begin{pmatrix} a & b_1 - ib_2 & c_1 - ic_2 \\ b_1 + ib_2 & d & e_1 - ie_2 \\ c_1 + ic_2 & d_1 + id_2 & -(a+d) \end{pmatrix}$

$\leadsto$ 8 free parameters

<u>strategy</u>: As for $su(2)$

$$T^1 = \begin{pmatrix} 0 & b_1 & 0 \\ b_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & -ib_2 & 0 \\ ib_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{T}^3 = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix}$$

$$T^4 = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ c_1 & 0 & 0 \end{pmatrix}, \quad T^5 = \begin{pmatrix} 0 & 0 & -ic_2 \\ 0 & 0 & 0 \\ ic_2 & 0 & 0 \end{pmatrix}$$

$$T^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & e_1 & 0 \end{pmatrix}, \quad T^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -ie_2 \\ 0 & ie_2 & 0 \end{pmatrix}, \quad \tilde{T}^8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & -d \end{pmatrix}$$

Replace $\tilde{T}^3, \tilde{T}^8$ w/ $T^3 = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$T^8 = \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & -2d \end{pmatrix}$$

in order to make $su(2) \subset su(3)$ manifest (first/last row). Not necessary but gives standard <u>Gell-Mann-matrices</u>.

<u>Normalization</u>: $\boxed{b_1 = \pm \frac{1}{2}}$ $\boxed{b_2 = \pm \frac{1}{2}}$ $\boxed{a = \pm \frac{1}{2}}$

$\boxed{c_1 = \pm \frac{1}{2}}$ $\boxed{c_2 = \pm \frac{1}{2}}$ $\boxed{d = \pm \frac{1}{2\sqrt{3}}}$

$\boxed{e_1 = \pm \frac{1}{2}}$ $\boxed{e_2 = \pm \frac{1}{2}}$

$$\Longrightarrow \underbrace{T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{su(2)}$$

$$T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^8 = \frac{1}{2}\left(\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\right)$$

$$\Longleftrightarrow T^a = \frac{\lambda^a}{2} \text{ w/ } \lambda^a \ldots \text{ Gell-Mann-matrices!}$$

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3) · Practical definition for Lie-algebras/groups

rank (Lie-group) = rank (Lie-algebra)

$= \dim(\underbrace{\text{Cartan subalgebra}}_{\text{maximal abelian subalgebra}})$

maximal abelian subalgebra

$\Longleftrightarrow$ maximal set of commuting generators

[Practical tip: for matrix groups its always the set of diagonal matrices]

· For SU(N) its easy to calculate the rank:

hermitian $\longrightarrow$ diagonal part real $\Longrightarrow$ N diagonal matrices

traceless $\longrightarrow$ constraint that removes one parameter

$\Longrightarrow$ rank$(SU(N)) = N-1$

· Casimir operator: Linear combination of symmetric homogeneous polynomials in the generators, that commutes with all elements of the algebra.

(Practical definition)

↳ Needs to be proportional to $\mathbb{1}$ to achieve this generally (Schur's Lemma)

↳ It can be shown that #Casimirs = rank(Lie-algebra)

$\boxed{SU(2)}$ Per definition: $\boxed{C_m = \kappa_{a_1 \dots a_m} T^{a_1} \dots T^{a_m} \quad \text{w/} \quad m = 1 \dots \text{rank}(su(N))}$

SU(2) has only a quadratic Casimir.

Again, per definition κ_{ab} must be a symmetric invariant tensor.

$\longrightarrow \kappa_{ab} = 2\,tr(T_a T_b) = \delta_{ab}$

↳ this normalization is arbitrary

$\Longrightarrow C_2 = (T^1)^2 + (T^2)^2 + (T^3)^2$

[This is the squared spin operator in QM: $S^2 = S_x^2 + S_y^2 + S_z^2$]

$\boxed{SU(3)}$ Same procedure!

Quadratic Casimir: $\boxed{C_1 = T_a T^a = \sum_{a=1}^{8} (T^a)^2}$

However, since the rank$(SU(3)) = 2$, there is also a cubic Casimir.

$\Longrightarrow$ Need to construct invariant & symmetric tensor out of three generators

Guess 1: $tr(T^a T^b T^c)$ not symmetric ↯

Guess 2: $tr(\{T^a, T^b\} T^c) \equiv d^{abc}$ ✓ again defined up to const.

$\Longrightarrow \boxed{C_2 = d_{abc} T^a T^b T^c}$

I. Representation: Groups do not have to only act on a space of cont. dim but can also act on ~~higher~~ other dim. spaces.

↳ e.g. $SU(2)$ does not have to act on two-comp. spinors but can also act on higher dim spinors

$\Longrightarrow$ def: Map $R: G \longrightarrow R(G) \subset GL(V)$ that preserves group structure

Think as follows: fund. commutation rel. of Lie algebra $[T^a, T^b] = i f^{abc} T^c$

Solutions to this in different dim. correspond to different reps.

The Casimir operators commute w/ all elements of the algebra per definition. Thus, they only depend on the representation

$\Longrightarrow$ Use their Eigenvalues to label reps!

Example $SU(2)$: Cartan subalgebra: S_z ; Casimir: $S^2 = S_x^2 + S_y^2 + S_z^2$ } maximal set of commuting operators

$S_z |\alpha\rangle = m_s |\alpha\rangle$

$S^2 |\alpha\rangle = s(s+1) |\alpha\rangle$ $\Longrightarrow |\alpha\rangle \equiv |s, m\rangle$

4) Fundamental representation (or defining rep)

Smallest-dimensional faithful rep.
(= injectiv)

→ For matrix Lie-groups this is the defining rep. or in other words the representation map is just the identity

Adjoint representation

Representation that acts on the Lie-algebra itself.

$\boxed{SU(N)}$ Fundamental rep

$U \in SU(N)$

$\Psi \to U\Psi$ for some $\Psi \in \mathbb{C}^N$ ~~vector of dim(V)=N~~

Adjoint rep ⇝ N-dim.

$\Phi \to U \Phi U^{-1}$ for some $U \in SU(N)$, $\Phi \in su(N)$

⇝ $\dim(su(N)) = N^2 - 1$

By using the exponential map we can rewrite it in terms of the generators as

$$T^a_{adj} \longrightarrow (\delta^a{}_c + i\theta_b i f^{ba}{}_c) T^c$$

or equivalently

$$\boxed{(T^a_{adj})_{bc} = -i f^a{}_{bc}}$$

Note: The group representations are constructed by computing the algebra representations and mapping them to the group via the exp. map.

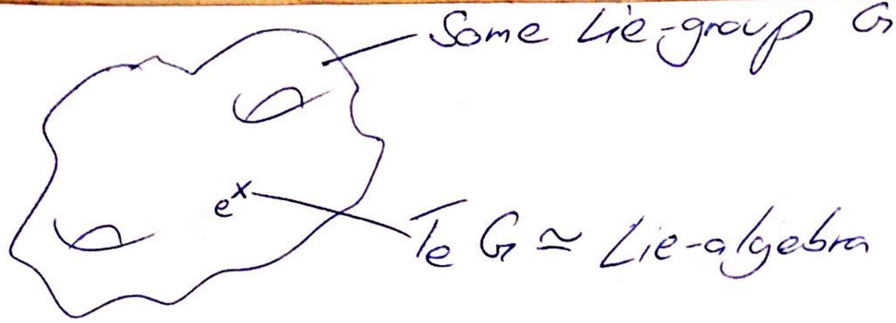
example: $SU(2)$

$|\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle \to e^{i\theta^a \frac{\sigma^a}{2}} |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle$ fund. rep: $T^a = \frac{\sigma^a}{2}$

$|1, {}^{+1}_{\;0}_{-1}\rangle \to e^{i\theta^a T^a_{adj}} |1, {}^{+1}_{\;0}_{-1}\rangle$ adj rep: $(T^a_{adj})_{bc} = i f^a{}_{bc}$

↳ $T^1_{adj} = i\begin{pmatrix} 0&0&0 \\ 0&0&-1 \\ 0&+1&0 \end{pmatrix}$, $T^2_{adj} = i\begin{pmatrix} 0&0&+1 \\ 0&0&0 \\ -1&0&0 \end{pmatrix}$, $T^3_{adj} = i\begin{pmatrix} 0&-1&0 \\ +1&0&0 \\ 0&0&0 \end{pmatrix}$ could be familiar from QM1 course (probably in different basis)

5)



simply calculating $[T^a, T^b] = i f^{abc} T^c$ one finds that for $SU(2)$ ~~SU(3)~~

$$f^{abc} = \varepsilon^{abc}$$

$\left(\text{As stated in 4)} \quad (T_{adj}^a)_{bc} = -i f^a{}_{bc} \overset{SU(2)}{=} -i \varepsilon^a{}_{bc} \right)$

and for $SU(3)$:

$$f^{123} = 1$$

$$f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2}$$

$$f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$

All others related by permutation or vanish.

2 Gauge Theories

1) global invariance obvious $\quad U \equiv e^{-ie\alpha}$

$$\mathcal{L} \to \mathcal{L}' = (U\psi)^\dagger \gamma^0 (i\not{\partial} - m) U\psi$$
$$= \psi^\dagger U^\dagger \gamma^0 (i\not{\partial}-m) U\psi$$
$$= \psi^\dagger \gamma^0 \underbrace{U^\dagger U}_{=\mathbb{1}} \gamma^0 (i\not{\partial}-m)\psi$$
$$= \bar{\psi}(i\not{\partial}-m)\psi = \mathcal{L}$$

Noether: $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ w/ ϕ some field

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\partial_\mu\phi$$
$$= \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right)}_{=0 \text{ Euler-Lagrange-equation}}\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right)$$

$\overset{!}{=} 0$ (or $\partial_\mu K^\mu$ for some fct K^μ but irrelevant here)

$\xrightarrow{\text{Noether}}$ $\boxed{j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi}$ w/ $\partial_\mu j^\mu = 0$

Dirac-$\mathcal{L}$: $j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\delta\psi = i\bar{\psi}\gamma^\mu(-ie\psi) = e\bar{\psi}\gamma^\mu\psi$

$$\psi' = e^{-ie\alpha}\psi = (1-ie\alpha)\psi = \psi - ie\alpha\psi =: \psi + \alpha\delta\psi$$

$$\partial_\mu j^\mu = e[\partial_\mu\bar{\psi}\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi]$$
$$= e[im\bar{\psi}\psi + (-i)\bar{\psi}m\psi] = 0$$

eom: $i\gamma^\mu\partial_\mu\psi = m\psi$

$-i\partial_\mu\bar{\psi}\gamma^\mu = m\bar{\psi}$

$$Q = \int d^3x\, j^0 = e\int d^3x\, \psi^\dagger\psi$$

2) $\bar{\Psi} m \Psi \to \bar{\Psi} U^{\dagger} m U \Psi = \bar{\Psi} m \Psi$ ✓

$\bar{\Psi} i \not{\partial} \Psi \to \bar{\Psi} U^{\dagger} i \not{\partial} (U \Psi) = \bar{\Psi} U^{\dagger} i U \not{\partial} \Psi + \bar{\Psi} U^{\dagger} i (\not{\partial} U) \Psi$

$= \bar{\Psi} i \not{\partial} \Psi + \bar{\Psi} i (-ie \partial_{\mu} \alpha \gamma^{\mu}) \Psi$

$= \bar{\Psi} i \not{\partial} \Psi + \underbrace{e \bar{\Psi} \not{\partial} \alpha \Psi}_{\equiv \delta \mathcal{L}_{\Psi}}$ $\Rightarrow$ not invariant!

3) $j^{\mu} A_{\mu} = e \bar{\Psi} \gamma^{\mu} A_{\mu} \Psi$

$\longrightarrow e \bar{\Psi} U^{\dagger} \gamma^{\mu} (A_{\mu} + \partial_{\mu} \alpha) U \Psi$

$= e \bar{\Psi} \gamma^{\mu} (A_{\mu} + \partial_{\mu} \alpha) \Psi$

$= e \bar{\Psi} \gamma^{\mu} A_{\mu} \Psi + \underbrace{e \bar{\Psi} \gamma^{\mu} \partial_{\mu} \alpha \Psi}_{\equiv \delta \mathcal{L}_A = \delta \mathcal{L}_{\Psi}}$

$\Longrightarrow \mathcal{L}_2 = \bar{\Psi} (i \not{\partial} - m) \Psi - j^{\mu} A_{\mu}$ is invariant

4) $D_{\mu} \Psi \longrightarrow U (D_{\mu} \Psi)$ per definition

$\Longrightarrow D_{\mu} = \partial_{\mu} + i e A_{\mu}$

Replacing ∂_{μ} w/ D_{μ} in $\mathcal{L}$ leads to $\mathcal{L}_2$, which is invariant

5) To arrive a $\mathcal{L}_{QED}$ we need a gauge inv. kinetic term for A_{μ}

$\Longrightarrow \mathcal{L}_{QED} = \bar{\Psi} (i \not{D} - m) \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

w/ $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$

eoms: $(i \gamma^{\mu} D_{\mu} - m) \Psi = 0$

$\partial_{\mu} F^{\mu\nu} = j^{\nu} = e \bar{\Psi} \gamma^{\nu} \Psi$

6) $Q = e \int_{\Omega} d^3x \, \Psi^{\dagger} \Psi \overset{\text{MW eom \& time-independent } A_{\mu}}{=} \int_{\Omega} d^3x \, \partial_i \underbrace{F^{i0}}_{E^i}$

$= \int_{\partial\Omega} d^2x \, \underline{\hat{n}} \cdot \underline{E} = \int_{\partial\Omega} d\underline{S} \cdot \underline{E}$ Gauß law

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