“QCD and Standard Model”
Problem Set 10

1. Kaon oscillations and CP violation

In this exercise you will see how Kaon decays led to the discovery of CP violation in nature. In the following, let's first assume that CP is conserved for the weak interactions.

a) Consider the neutral particle
$$|K^0\rangle = |d\bar{s}\rangle \ ,$$
called Kaon, and its anti-particle
$$|\bar{K}^0\rangle = |\bar{d}s\rangle \ ,$$
where d and s are the down and strange quarks, respectively. Argue that, although the Kaon can be produced by strong interactions, it can only decay via the weak interaction.

b) Using the fact that Kaons are pseudoscalars, show how they transform under CP.

c) Use this to write two linear combinations of $|K^0\rangle$ and $|\bar{K}^0\rangle$, which are eigenstates of CP, with eigenvalues +1 and -1.

d) We know from observations that K^0 can decay into two pions, $\pi^+\pi^-$ or $\pi^0\pi^0$. Using CP conservation, determine which of the above combinations participates in this decay.

e) We also know from observations that K^0 can decay into three pions $\pi^0\pi^0\pi^0$. Which combination participates in this decay ?

Since the decay into 3 pions is much less probable than the decay into 2 pions, the above linear combinations are sometimes called K_S^0 and K_L^0, where one of them is ”short-lived” and the other is ”long-lived”. However, in 1964 it was discovered that the ”long-lived” state could indeed decay into 2 pions, around 0.2% of the time. This was the first indication that CP is violated in weak interactions. So the ”real” short-lived and long-lived particles are

$$|K_S^0\rangle = N(p\,|K^0\rangle + q\,|\bar{K}^0\rangle) \ ,$$
$$|K_L^0\rangle = N(p\,|K^0\rangle - q\,|\bar{K}^0\rangle) \ ,$$

where $N = (|p|^2 + |q|^2)^{-1/2}$ is a normalization factor. Due to CP violation the (complex) numbers p and q are not equal.

f) Finally, we want to better understand the phenomenon of $K - \bar{K}$ *oscillations*. Draw the lowest-order Feynman diagrams responsible for the process $K^0 \leftrightarrow \bar{K}^0$.

g) The dynamics of this process can be effectively captured by the following equation
$$i\frac{d}{dt}\begin{pmatrix}|K^0\rangle \\ |\bar{K}^0\rangle\end{pmatrix} = \begin{pmatrix} m - i/2\Gamma & -p^2 \\ -q^2 & m - i/2\Gamma\end{pmatrix}\begin{pmatrix}|K^0\rangle \\ |\bar{K}^0\rangle\end{pmatrix} \ ,$$
where the 2×2-matrix may be understood as an ”effective” Hamiltonian with Γ the decay rate, modeling the decay into the pions. Find the eigenstates and the eigenvalues of this Hamiltonian. Explain why $|K_S^0\rangle$ and $|K_L^0\rangle$ have different masses and different life-times.

2. A specific CKM Matrix and its independent parameters

Assume that the mass matrices for the up- and down- type quarks have the following forms (in the basis of weak interaction eigenstates)

$$M^{(u)} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} , \qquad \text{and} \qquad M^{(d)} = m \begin{pmatrix} 1+a^2 & ab & 0 \\ ab & 1+b^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

respectively. Here $m_i, [i = u, c, t]$ the mass of the respective quark flavor, m a parameter with dimensions of mass, and a, b real.

a) Find the CKM matrix. How many independent parameters does it have? Parametrize them in terms of a and b.

b) Will there be a physical CP-violating phase? Explain.

3. A CKM matrix in the case of two generations

Let us now restrict ourselves to two generations of quarks. Take the mass matrix of the up-type quarks to be diagonal, and the one for the down-type quarks to be the following

$$M^{(d)} = m \begin{pmatrix} 0 & a \\ a & 2b \end{pmatrix} ,$$

with m a parameter with dimensions of mass and a, b real with $a \ll b$.

a) Find the 2×2 analog of the CKM matrix in terms of a and b.

b) Take $m_s/m_d \approx 20$ and compare the value of the mixing angle θ_{mix} with its experimentally measured value $\theta_{\text{mix}} \approx 13^o$.