"QCD and Standard Model"
Problem Set 3

Non-Abelian gauge theories

Non-Abelian gauge theories play an important role in particle physics. In this exercise we shall discuss some properties of non-Abelian gauge theories with group SU(N).

a) The Yang-Mills Lagrangian density for a non-Abelian gauge field A^a_μ can be written in the following form,

$$\mathcal{L}_{\rm YM} \;=\; -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \,,$$

with $F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, f^{abc} the structure constants of SU(N) and summation over all repeated indexes is tacitly assumed. Show that the above can be equivalently written as

$$\mathcal{L}_{\rm YM} = -\frac{1}{2} {\rm Tr}\left(F_{\mu\nu} F^{\mu\nu} \right) \,,$$

where $F_{\mu\nu} \equiv F^a_{\mu\nu} T^a$, with T^a the generators of SU(N).

b) Consider now

$$\mathcal{L} = \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm D} \,,$$

where the generalized Dirac Lagrangian reads

$$\mathcal{L}_{\rm D} = \bar\psi (i\gamma^\mu D_\mu)\psi - m\bar\psi\psi \,,$$

with the covariant derivative $D_\mu \equiv \partial_\mu - igA_\mu$, and $A_\mu \equiv A^a_\mu T^a$. Show that $\mathcal{L}$ is invariant under the Yang-Mills transformation

$$\begin{aligned} A_\mu(x) \quad &\longrightarrow \quad A'_\mu(x) \;=\; U(x) A_\mu(x) U^{-1}(x) - \frac{i}{g}\left[\partial_\mu U(x)\right] U^{-1}(x) \,, \\ \psi(x) \quad &\longrightarrow \quad \psi'(x) \;=\; U(x)\psi(x) \,, \end{aligned}$$

for any SU(N) matrix $U(x)$.

c) Take a global SU(N) tranformation, i.e. $U = const.$ and find the conserved current using the Noether theorem. Compare the corresponding charge to the one you found in the Abelian case in problem set 1.

d) Derive the equations of motion for A^a_μ following from $\mathcal{L}$ and compare them with the corresponding equations of motion for an Abelian gauge field A_μ in QED.

e) Show that

$$[D_\mu, D_\nu]\psi \;=\; -igF_{\mu\nu}\psi \,,$$

where the brackets $[\ldots]$ stand for the commutator.

f) Demonstrate the *Bianchi identity*

$$D_\rho F_{\mu\nu} + D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} \;=\; 0 \,.$$

g) Consider now the dual field strength tensor defined as

$$\tilde F_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}.$$

First, show that in the Abelian case

$$F^{\mu\nu}\tilde{F}_{\mu\nu} = \partial_\mu K^\mu \,,$$

and find K_μ. Does this term have any consequences for the equations of motion ?
Next, show that the corresponding quantity in the non-Abelian case (which has to be SU(N)-invariant) is

$$\mathrm{Tr}\left(F^{\mu\nu}\tilde{F}_{\mu\nu}\right) = \partial_\mu \tilde{K}^\mu \,.$$

What is $\tilde{K}_\mu$ now ?

Discuss whether a term like $\theta\,\mathrm{Tr}\left(F^{\mu\nu}\tilde{F}_{\mu\nu}\right)$, with θ a constant, is allowed in the Lagrangian. Does it appear in the equations of motion ? What are the implications ?