https://www2.physik.uni-muenchen.de/lehre/vorlesungen/sose_22/TMP-TA3/index.html

Sheet 5:

Hand-out: Friday, May 27, 2022[1]

Problem 1 Goldstone mode in the Heisenberg ferromagnet

In this problem we discuss an example of spontaneous symmetry breaking in the ground state of the one-dimensional Heisenberg model, and show that it features a non-relativistic gapless Goldstone mode.

(1.a) As a warmup, consider the *classical* 1D Heisenberg ferromagnet ($J < 0$), with the classical energy

$$E = J \sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}. \tag{1}$$

Find all classical ground state configurations $\{\boldsymbol{S}_j\}$ which minimize the energy functional $E[\{\boldsymbol{S}_j\}]$ and determine the ground state energy E_0. Show that E is invariant under global $O(3)$ rotations. Are the ground states minimizing $E[\{\boldsymbol{S}_j\}]$ symmetric under $O(3)$?

(1.b) Now we move on the the *quantum* 1D Heisenberg ferromagnet ($J < 0$), with the Hamiltonian

$$\hat{\mathcal{H}} = J \sum_j \hat{\boldsymbol{S}}_j \cdot \hat{\boldsymbol{S}}_{j+1}. \tag{2}$$

Using the variational principle, show that the classical ground states $|\{\boldsymbol{\sigma}_j\}\rangle$, obtained by multiplying the positive-eigenvalue eigenstates of $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{S}}_j$, are true ground (and thus eigen-) states of $\hat{\mathcal{H}}$.

(1.c) Consider again the *quantum* 1D Heisenberg ferromagnet ($J < 0$) from (1.b). Choose the classical ground state $|\mathrm{FM}_z\rangle$ with all spins pointing along z and define the following set of all states with total magnetization $S^z_{\mathrm{tot}} = L/2 - 1$,

$$\{\hat{S}^-_j |\mathrm{FM}_z\rangle\}_{j=1\ldots L}. \tag{3}$$

Show that the Hamiltonian $\hat{\mathcal{H}}$ is block-diagonal in $S^z_{\mathrm{tot}} = \sum_j \hat{S}^z_j$ and diagonalize the block with $S^z_{\mathrm{tot}} = L/2 - 1$. Show that the resulting one-magnon states have a dispersion relation

$$\omega_k = -J\left(1 - \cos(k_x)\right) \simeq -\frac{J}{2} k_x^2 + \mathcal{O}(k_x^4). \tag{4}$$

This is the gapless (non-relativistic) Goldstone mode of this model.

[1] If you would like to present your solution(s), feel free to send them to Henning Schlömer until Fri, June 03.

Problem 2 Jordan-Wigner transformation

In the lecture we solve the 1D XY model by mapping it to a free-fermion Hamiltonian. In this problem we consider the XXZ Hamiltonian,

$$\hat{\mathcal{H}}_{\text{XXZ}} = -J_\perp \sum_j \left(\hat{S}^x_{j+1} \hat{S}^x_j + \hat{S}^y_{j+1} \hat{S}^y_j \right) - J_z \sum_j \hat{S}^z_{j+1} \hat{S}^z_j, \tag{5}$$

where $\hat{S}^\mu_j = \hat{\sigma}^\mu_j \hbar/2$ is a spin-$1/2$ operator.

(2.a) Apply the Jordan-Wigner transformation to derive an equivalent Hamiltonian to $\hat{\mathcal{H}}_{\text{XXZ}}$ expressed in terms of spin-less Jordan-Wigner fermions. Write out the interactions in momentum modes and show that the resulting Hamiltonian becomes

$$\hat{\mathcal{H}}_{\text{XXZ}} = \sum_k \omega_k \hat{c}^\dagger_k \hat{c}_k - \frac{J_z}{L} \sum_{k,k',q} \cos(q) \hat{c}^\dagger_{k-q} \hat{c}^\dagger_{k'+q} \hat{c}_{k'} \hat{c}_k \tag{6}$$

where L is the total number of lattice sites (assume periodic boundary conditions), and

$$\omega_k = J_z - J_\perp \cos k. \tag{7}$$

(2.b) Assume $J_z = J_\perp = J > 0$ and describe the ground state and low-energy excitations in terms of Jordan-Wigner fermions. How does the state relate to the Heisenberg ferromagnet discussed in Problem 1?

(2.c) Assume $J_z = 0$ (as in the lecture, XY model) and describe the low-energy excitations of the model. Does the model have a gapless low-energy mode?

(2.d) Assume now $J_\perp = 0$ (Ising model). Does the model have a gapless low-energy mode?

Problem 3 Attaching strings

Here we study hard-core particles $\hat{d}^{(\dagger)}_j$ on the sites of a one-dimensional chain, coupled to additional spin-$1/2$ particles $\hat{\boldsymbol{\sigma}}_{\langle i,j\rangle}$ on the links $\langle i,j\rangle$ of the chain. The system is described by the Hamiltonian:

$$\hat{\mathcal{H}} = -t \sum_j \left(\hat{d}^\dagger_{j+1} \hat{\sigma}^z_{\langle j+1,j\rangle} \hat{d}_j + \text{h.c.} \right) - \mu \sum_j \hat{d}^\dagger_j \hat{d}_j \tag{8}$$

(this is a so-called $\mathbb{Z}_2$ lattice gauge theory).

(3.a) Assume first that the particles $\hat{d}_j \equiv \hat{a}_j$ are bosons, $[\hat{a}_i, \hat{a}^\dagger_j] = \delta_{i,j}$. Introduce new operators $\hat{\alpha}_i = \hat{a}_i \prod_{j>i} \hat{A}_j$, with an appropriately chosen string of operators $\hat{A}_j$, such that the system Hamiltonian can be written as a free boson model:

$$\hat{\mathcal{H}}_a = -t \sum_j \left(\hat{\alpha}^\dagger_{j+1} \hat{\alpha}_j + \text{h.c.} \right) - \mu \sum_j \hat{\alpha}^\dagger_j \hat{\alpha}_j. \tag{9}$$

Assume an infinite system for simplicity.

(3.b) Now assume that the particles $\hat{d}_j \equiv \hat{c}_j$ are fermions, $\{\hat{c}_i, \hat{c}^\dagger_j\} = \delta_{i,j}$. Find new operators $\hat{\eta}_i = \hat{c}_i \prod_{j>i} \hat{B}_j$ with appropriate $\hat{B}_j$, such that a free-fermion Hamiltonian can be obtained. Assume an infinite system again.