Sheet 2:

Hand-out: Friday, May 06, 2022[1]

Problem 1 Yastrow variational states and Laughlin wavefunctions

A popular family of variational wavefunctions is defined by the Yastrow wavefunctions, which take the form:

$$\Psi(z_1, ..., z_N) = \prod_{i<j} \phi_2(z_i - z_j) \;\times\; \prod_{j=1}^{N} \phi_1(z_j), \tag{1}$$

where z_j are general (possibly vector-valued) coordinates, $i, j = 1...N$ and $\phi_1(z)$ and $\phi_2(z)$ are arbitrary single-particle and pair-wavefunctions.

(1.a) For normalized many-body wavefunctions $|\Psi\rangle$, show that the density expectation value is

$$n(x) = \langle\Psi|\hat{\Psi}^\dagger(x)\hat{\Psi}(x)|\Psi\rangle = N \int dz_2...dz_N \; |\Psi(x, z_2, ..., z_N)|^2, \tag{2}$$

and the two-point correlation function is

$$\begin{aligned} g^{(2)}(x_1, x_2) &= \langle\Psi|\hat{\Psi}^\dagger(x_1)\hat{\Psi}(x_1)\hat{\Psi}^\dagger(x_2)\hat{\Psi}(x_2)|\Psi\rangle \\ &= N(N-1) \int dz_3...dz_N \; |\Psi(x_1, x_2, z_3, ..., z_N)|^2. \end{aligned} \tag{3}$$

(1.b) Use the results from (3.a) to show for the Yastrow ansatz that

$$n(x) \propto |\phi_1(x)|^2 \tag{4}$$

and

$$g^{(2)}(x_1, x_2) \propto |\phi_2(x_1 - x_2)|^2 |\phi_1(x_1)|^2 |\phi_1(x_2)|^2. \tag{5}$$

(You do not have to evaluate the integrals explicitly, just state them!)

(1.c) A case of particular relevance corresponds to 2D spin-polarized fermions in a magnetic field, where $z_j = x_j + iy_j$ are complex variables describing their coordinates x_j, y_j in the 2D plane. The low-energy single-particle orbitals in this problem are labeled by $m = 0, 1, 2, 3, ...$ and (up to normalization) given by

$$\psi_m(z) = z^m e^{-|z|^2/4}. \tag{6}$$

Calculate the fermionic Slater determinant state obtained when all states $m = 0, ..., N$ are filled with one fermion and show that, up to normalization, it takes the form

$$\Psi_{\mathrm{F}}(z_1, ..., z_N) = \prod_{i<j} (z_i - z_j) \; \exp\left[-\sum_{j=1}^{N} |z_j|^2/4\right]. \tag{7}$$

This wavefunction has the Yastrow form; it's the so-called Vandermonde determinant.

[1]If you would like to present your solution(s), feel free to send them to Felix Palm until Fri, May 13.

(1.d) Construct a *bosonic* Yastrow-type wavefunction $\Psi_{\rm B}(z_1, ..., z_N)$ with $\phi_1(z) = \exp(-|z|^2/4)$ and according to the following rules: (i) $\phi_2(z)$ is a polynomial of only z (not the complex conjugate z^*); (ii) the state $\Psi_{\rm B}(z_1, ..., z_N)$ must have zero energy when point-like interactions

$$\hat{\mathcal{H}}_{\rm int} = \frac{1}{2}\sum_{i\neq j} g\delta(z_i - z_j) \tag{8}$$

are considered (you can use results from 3.b here); (iii) among all Yastrow wavefunctions satisfying (i) and (ii), find the one with minimal powers of z_n.

The result you obtain here is the famous *Laughlin wavefunction*.

Problem 2 Solving problems in second quantization formalism

In this problem we study a non-interacting hopping problem using the formalism of second quantization. Consider a bosonic or fermionic field $\hat{\psi}_j$ defined on lattice sites $j \in \mathbb{Z}$ on an infinite one-dimensional chain. The Hamiltonian of the system is:

$$\hat{\mathcal{H}} = -\sum_j \Big(t_1 \ \hat{\psi}^\dagger_{j+1}\hat{\psi}_j + t_2 \ \hat{\psi}^\dagger_{j+2}\hat{\psi}_j + \text{h.c.}\Big). \tag{9}$$

(2.a) Define the Fourier-transformed field $\hat{\psi}(k)$ in momentum space and derive it's (anti-) commutation relations for (fermionic) bosonic fields $\hat{\psi}_j$. On which interval is k defined?

(2.b) Express $\hat{\psi}_j$ in terms of the new field $\hat{\psi}(k)$ and insert this result into the Hamiltonian. Show that the Hamiltonian simplifies and becomes an integral over uncoupled momentum modes.

(2.c) Describe the eigenstates of the Hamiltonian using your result in (2.b).

Problem 3 Second quantization

In this problem we study more examples of the second quantization formalism.

(3.a) Consider bosonic (fermionic) fields $\hat{\psi}_m$ describing single-particle orbitals $|\psi_m\rangle$ labeled by m, and obeying the canonical (anti-) commutation relations

$$[\hat{\psi}_n, \hat{\psi}^\dagger_m] = \delta_{m,n}, \qquad \Big(\{\hat{\psi}_n, \hat{\psi}^\dagger_m\} = \delta_{m,n} \Big). \tag{10}$$

Consider a unitary transformation U of the single-particle orbitals $|\psi_m\rangle$, transforming them to a new single-particle basis $|\phi_m\rangle = U|\psi_m\rangle = \sum_n U_{n,m}|\psi_n\rangle$, with $\sum_n U^*_{m',n}U_{n,m} = \delta_{m',m}$, i.e. $U^\dagger U = 1$. Show that the new 2nd-quantized field operators

$$\hat{\phi}^\dagger_m = \sum_n U_{n,m}\hat{\psi}^\dagger_n \tag{11}$$

satisfy the same canonical (anti-) commutation relations

$$[\hat{\phi}_n, \hat{\phi}^\dagger_m] = \delta_{m,n}, \qquad \Big(\{\hat{\phi}_n, \hat{\phi}^\dagger_m\} = \delta_{m,n} \Big). \tag{12}$$

(3.b) Consider a point-like three-particle interaction, taking the first-quantized form

$$\hat{\mathcal{H}} = \frac{g_3}{6} \sum_{i \neq j \neq k \neq i} \delta(\hat{\boldsymbol{r}}_i - \hat{\boldsymbol{r}}_j)\delta(\hat{\boldsymbol{r}}_j - \hat{\boldsymbol{r}}_k), \tag{13}$$

and construct the corresponding Hamiltonian in second quantization.

(3.c) In the lecture we constructed the 2nd quantized interaction operator

$$\hat{U} = \int d^d\boldsymbol{r}\ d^d\boldsymbol{r}'\ \hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}^\dagger(\boldsymbol{r}')\ \frac{1}{2}U(\boldsymbol{r} - \boldsymbol{r}')\ \hat{\psi}(\boldsymbol{r}')\hat{\psi}(\boldsymbol{r}) \tag{14}$$

corresponding to the 1st quantized interaction potential $U(\boldsymbol{r})$ between two particles at distance $\boldsymbol{r}$. Show by an explicit calculation that the action of $\hat{U}$ on the many-body state $|\boldsymbol{r}_1, ..., \boldsymbol{r}_N\rangle$ is:

$$\hat{U}|\boldsymbol{r}_1, ..., \boldsymbol{r}_N\rangle = \left(\frac{1}{2}\sum_{i \neq j} U(\boldsymbol{r}_i - \boldsymbol{r}_j)\right)|\boldsymbol{r}_1, ..., \boldsymbol{r}_N\rangle. \tag{15}$$

(3.d) Write the interaction $\hat{U}$ from Eq. (14) in terms of 2nd quantized momentum operators $\hat{\psi}^\dagger_{\boldsymbol{k}}$. To simplify the final expression, express $U(\boldsymbol{r})$ in terms of its Fourier transform $\tilde{U}(\boldsymbol{q})$.