AKLT model: Exercise

The ground state of the AKLT model has the following MPS form:

$$|\psi\rangle = |\vec{\sigma}_N\rangle \, \mathrm{Tr}[B^{\sigma_1} \ldots B^{\sigma_N}], \tag{1}$$

where

$$B^{+1} = \sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; B^{0} = \sqrt{\frac{1}{3}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; B^{-1} = \sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{2}$$

(a) Verify that B matrices are right-normalized.
(b) Compute the transfer operator $T = \sum_{\sigma_\ell} B^{\dagger\alpha'_{\ell+1}}{}_{\sigma_\ell \alpha'_\ell} B^{\alpha_\ell \sigma_\ell}{}_{\alpha_{\ell+1}}$, and verify that its eigenvalues are $1, -1/3, -1/3, -1/3$.
(c) Compute the transfer operator for $\hat{S}^z$, $\hat{S}^{\pm,}$ and $e^{i\pi \hat{S}^z}$, using

$$T_{\hat{O}_{[\ell]}} = \sum_{\sigma_\ell, \sigma'_\ell} B^{\dagger\alpha'_{\ell+1}}{}_{\sigma'_\ell \alpha'_\ell} O^{\sigma'_\ell}{}_{\sigma_\ell} B^{\alpha_\ell \sigma_\ell}{}_{\alpha_{\ell+1}}. \tag{3}$$

(d) Compute the asymptotic ($\lim_{|\ell-\ell'|\to\infty} \lim_{N\to\infty}$) behavior of

$$\chi^{zz}_{\ell\ell'} \equiv \langle\psi|\hat{S}^z_{[\ell]} \hat{S}^z_{[\ell']}|\psi\rangle \tag{4a}$$

$$\chi^{+-}_{\ell\ell'} \equiv \langle\psi|\tfrac{1}{2}\hat{S}^+_{[\ell]} \tfrac{1}{2} S^-_{[\ell']}|\psi\rangle \tag{4b}$$

$$\chi^{\mathrm{string}}_{\ell\ell'} \equiv \langle\psi|\hat{S}^z_{[\ell]} (\Pi_{\bar{\ell}=\ell+1}^{\ell'-1} e^{i\pi \hat{S}^z_{[\bar{\ell}]}}) \hat{S}^z_{[\ell']}|\psi\rangle\,. \tag{4c}$$

Check your results: $\chi^{zz}_{\ell\ell'} \sim e^{-|\ell-\ell'|/\xi}$ with $\xi = \frac{1}{\ln 3}$, and $\chi^{\mathrm{string}}_{\ell\ell'} \sim -\frac{4}{9}$.