Problem Set 8.

a) $i \sum_j \bar{\psi}_j \underbrace{\not{D}}_{\gamma_\mu D^\mu} \psi_j$ $\qquad$ j = particle species

$\psi \in \{ \underset{\overset{\shortparallel}{\binom{u}{d}_L}}{Q_L}, u_R, d_R, \underset{\overset{\shortparallel}{\binom{e}{\nu}_L}}{E_L}, e_R \}$

Let's look at the interaction terms that contain W_μ^3 and B_μ:

$$\mathcal{L}_{int} \supseteq i \bar{Q}_L^j \gamma^\mu \left(-i g W_\mu^3 \tau^3 - i g' \frac{y}{2} B_\mu \right) Q_L^j \quad \text{(sum over families)}$$
$$+ \bar{E}_L^j \gamma^\mu \left(g W_\mu^3 \tau^3 + g' \frac{y}{2} B_\mu \right) E_L^j$$
$$+ \bar{u}_R^j \gamma^\mu g' \frac{y}{2} B_\mu u_R^j$$
$$+ \bar{d}_R^j \gamma^\mu g' \frac{y}{2} B_\mu d_R^j$$
$$+ \bar{e}_R^j \gamma^\mu g' \frac{y}{2} B_\mu e_R^j$$

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_\mu^1 \mp W_\mu^2 \right)$$

$$Z_\mu = \frac{1}{\sqrt{g^2+g'^2}} \left(g W_\mu^3 - g' B_\mu \right) \equiv \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu$$

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}} \left(g' W_\mu^3 + g B_\mu \right) \quad \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu$$

$$\theta_W \Rightarrow \quad \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}} \leftrightarrow \tan\theta_W = \frac{g'}{g}$$

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}}_{R(\theta_W)} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}$$

$$\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} = R^T(\theta_W) \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}$$

$$\Rightarrow \mathcal{L}_{int} = Z_\mu \Big[\bar{Q}_L^j \gamma^\mu (\underbrace{g\cos\theta_W}_{e} \tau^3 - \underbrace{g' \tfrac{y}{2} \sin\theta_W}_{= e\, y/2}) Q_L^j$$
$$+ \bar{E}_L^j \gamma^\mu (g\cos\theta_W \tau^3 - g' \tfrac{y}{2} \sin\theta_W) E_L^j$$
$$- \bar{u}_R^j \gamma^\mu g' \tfrac{y}{2} \sin\theta_W u_R^j$$
$$- \bar{d}_R^j \gamma^\mu g' \tfrac{y}{2} \sin\theta_W d_R^j$$
$$- \bar{e}_R^j \gamma^\mu g' \tfrac{y}{2} \sin\theta_W e_R^j \Big]$$
$$+ A_\mu \Big[\bar{Q}_L^j \gamma^\mu (g\sin\theta_W \tau^3 + g' \tfrac{y}{2} \cos\theta_W) Q_L^j$$
$$+ \bar{E}_L^j \gamma^\mu (g\sin\theta_W \tau^3 + g' \tfrac{y}{2} \cos\theta_W) E_L^j$$
$$+ \bar{u}_R^j \gamma^\mu g' \tfrac{y}{2} \cos\theta_W u_R^j$$
$$+ \bar{d}_R^j \gamma^\mu g' \tfrac{y}{2} \cos\theta_W d_R^j$$
$$+ \bar{e}_R^j \gamma^\mu g' \tfrac{y}{2} \cos\theta_W e_R^j \Big]$$

$$e \equiv g\sin\theta_W = g' \cos\theta_W .$$

Recall: $\boxed{Q = \tau^3 + \frac{y}{2}}$

$$Q\Psi = q\,\Psi \quad (\tau^3\Psi = 0 \text{ for singlets under } SU(2))$$

$$\Rightarrow \mathcal{L}_{int} = g Z_\mu j^{\mu, Z} + e A_\mu j^{\mu, E.M.}$$

$$\boxed{j_\mu^Z = \cos\theta_W \, j_\mu^3 - \sin\theta_W \tan\theta_W \, j_\mu^y}$$

$$j_\mu^3 = \underbrace{\bar{Q}_L^j \gamma_\mu \tau^3 Q_L^j}_{= (\bar{u}_L \, \bar{d}_L) \gamma_\mu \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_L \\ d_L \end{pmatrix}} + \bar{E}_L^j \gamma_\mu \tau^3 E_L^j$$

$$j_\mu^{Y} = \bar{Q}_L \gamma_\mu \frac{y_Q}{2} Q_L^j + \bar{E}_L^j \gamma_\mu \frac{y_E}{2} E_L^j$$

$$+ \bar{u}_R^j \gamma_\mu \frac{y_u}{2} u_R^j + \bar{d}_R^j \gamma_\mu \frac{y_d}{2} d_R^j$$

$$+ \bar{e}_R^j \gamma_\mu \frac{y_e}{2} e_R^j$$

$$y_Q = \frac{1}{3}, \quad y_E = -1, \quad y_u = \frac{4}{3}, \quad y_d = -\frac{2}{3}, \quad y_e = -2$$

$$j_\mu^{EM} = \bar{Q}_L^j \gamma_\mu Q\, Q_L^j + \bar{E}_L^j \gamma_\mu Q\, E_L^j$$

$$+ \bar{u}_R^j \gamma_\mu Q\, u_R^j + \bar{d}_R^j \gamma_\mu Q\, d_R^j$$

$$+ \bar{e}_R^j \gamma_\mu Q\, e_R^j$$

$$= \frac{2}{3} \bar{u}_L^j \gamma_\mu u_L^j - \frac{1}{3} \bar{d}_L^j \gamma_\mu d_L^j - \bar{e}_L^j \gamma_\mu e_L^j$$

$$+ \frac{2}{3} \bar{u}_R^j \gamma_\mu u_R^j - \frac{1}{3} \bar{d}_R \gamma_\mu d_R^j - \bar{e}_R^j \gamma_\mu e_R^j$$

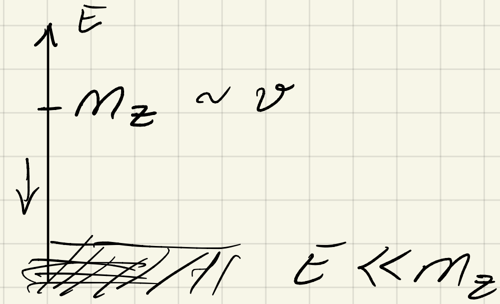
$$Q_{u_{L,R}} = \frac{2}{3}, \quad Q_{d_{L,R}} = -\frac{1}{3}, \quad Q_{e_{L,R}} = -1, \quad Q_{\nu_L} = 0.$$

To check this, take for instance

$$Q E_L = Q\begin{pmatrix} e_L \\ \nu_L \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e_L \\ \nu_L \end{pmatrix} \Rightarrow \quad Q e_L = -1$$

$$Q \nu_L = 0$$

b) Integrating out Z_μ:



Recall: The Kinetic terms for Z_μ, A_μ are contained in:

$$-\frac{1}{4} W^a_{\mu\nu} W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

$$= -\frac{1}{4}(\partial_\mu W^3_\nu - \partial_\nu W^3_\mu)^2 - \frac{1}{4} B^2_{\mu\nu} + (\text{Kin. terms for } W^{1,2}_\mu)$$

$$+ (ig f^{abc} \underbrace{W^b_\mu W^c_\nu}_{\text{self-interactions} \to \mathcal{O}(g^2)})^2$$

$$\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} = R^T \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = R^T_{ij} N^j_\mu, \text{ where } \begin{matrix} N^1_\mu \equiv Z_\mu \\ N^2_\mu = A_\mu \end{matrix}$$

$$\hookrightarrow \mathcal{L}_{Kin} = -\frac{1}{4}\left(\partial_\mu (R^T_{1j} N^j_\nu) - \partial_\nu (R^T_{1j} N^j_\mu)\right)^2$$

$$-\frac{1}{4}\left(\partial_\mu (R^T_{2j} N^j_\nu) - \partial_\nu (R^T_{2j} N^j_\mu)\right)^2$$

$$= \underbrace{(R^T_{1j} R^T_{1K} + R^T_{2j} R^T_{2K})}_{(R^T R)_{jK} = \delta_{jK}} \left(-\frac{1}{4}\right) (\partial_\mu N^j_\nu - \partial_\nu N^j_\mu) \times (\partial^\mu N^{K\nu} - \partial^\nu N^{K\mu})$$

$$= -\frac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)^2 = -\frac{1}{4}\Big[Z^2_{\mu\nu}$$

$$-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \qquad - F^2_{\mu\nu} \Big]$$

• Mass-term: $\frac{1}{2} m_Z^2 Z_\mu Z^\mu \subset (D_\mu H)^\dagger (D^\mu H)$

↳ EOM for Z_μ: $\partial_\mu Z^{\mu\nu} + m_Z^2 Z^\nu = -g j^{Z\nu}$ + ("Self-Interaction" $\mathcal{O}(g^2/m_Z^6)$)

(for A_μ): → $\partial_\mu F^{\mu\nu} = j^\nu$

$$\text{"} Z_\mu \sim \frac{1}{\Box + m_Z^2} \, (-g \, j^{Z\nu}) \text{"}$$

$$Z_\mu = g \int d^4 x' \int \frac{d^4 K}{(2\pi)^4} e^{-iK(x-x')} \times$$

$$\times \frac{g_{\mu\nu} - \frac{K_\mu K_\nu}{m_Z^2}}{K^2 - m_Z^2} j^{Z\nu}(x')$$

for $K^2 \ll m_Z^2 \Rightarrow Z_\mu \sim \frac{1}{m_Z^2} \times j^Z_\mu$

$\left(Z_\mu^4 \sim \mathcal{O}\left(\frac{j^{Z^4}}{m_Z^8}\right)\right)$

↳ $\frac{1}{2} m_Z^2 Z_\mu Z^\mu + g Z^\mu(x) j^Z_\mu$

↓

$$= \frac{1}{2} m_Z^2 g^2 \int \frac{d^4 K \, d^4 q}{(2\pi)^8} e^{-ix(K+q)} \left(\frac{g_{\mu\nu} - \frac{K_\mu K_\nu}{m_Z^2}}{K^2 - m_Z^2} \right) \left(\frac{g^{\mu\alpha} - q^\mu q^\alpha}{q^2 - m_Z^2} \right)$$

$$\times j^{Z\nu}(K) \, j^Z_\alpha(q)$$

$$+ g^2 \int \frac{d^4 K}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} e^{-ix(K+q)} j^Z_\mu(K) \left(\frac{g^{\mu\nu} - \frac{K^\mu K^\nu}{m_Z^2}}{K^2 - m_Z^2} \right) j^Z_\nu(q)$$

for $K^2 \ll m_Z^2$

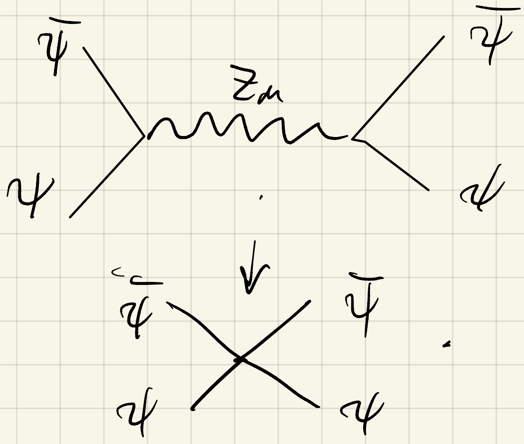
$$\mathcal{L}_{eff} = \frac{1}{2} \frac{g^2}{m_Z^2} g_{\mu\nu} g^{\mu\alpha} j^{Z\nu}(x) \, j^Z_\alpha(x)$$

$$- \frac{g^2}{m_Z^2} j^Z_\mu(x) \, j^Z_\nu(x) \, g^{\mu\nu}$$

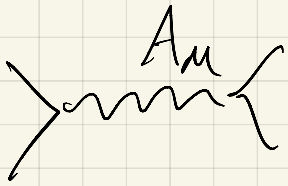
$$= -\frac{1}{2}\,\frac{g^2}{m_Z^2}\, j_\mu^Z(x)\, j^{Z\mu}(x)$$

Note: $Z_{\mu\nu}^2 \sim \frac{K^2}{m_Z^2} \times \frac{1}{m_Z^2} \ll \frac{1}{m_Z^2}$

$$j_\mu \sim \bar{\Psi}\,\gamma_\mu\,\Psi$$



$$\sim (\bar{\Psi}\gamma_\mu\Psi)(\bar{\Psi}\gamma^\mu\Psi)$$

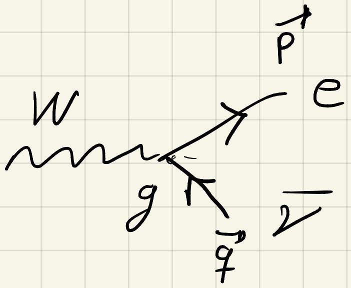
c) 

$m_A = 0 \Rightarrow$ The photon is massless, so it will be always produced, no matter how low the energies involved.

d) The neutral current is flavour diagonal, in both flavour basis and mass basis.

2) W boson Decay

a) $W \longrightarrow e\bar{\nu}$



The matrix element reads:

$$\mathcal{M} = \frac{g}{\sqrt{2}} \varepsilon_\mu \left(\bar{e}(\vec{p}) \gamma_\mu L \nu(\vec{q})\right) \, (*), \quad L = \frac{1}{2}(1+\gamma_5)$$

$$\mathcal{M}^+ = \frac{g}{\sqrt{2}} \varepsilon_\mu^* \nu^+(\vec{q}) L^+ \gamma_\mu^+ \bar{e}^+(\vec{p})$$

$$= \frac{g}{\sqrt{2}} \varepsilon_\mu^* \nu^+(\vec{q}) \underbrace{L^+}_{=L} \gamma_\mu^+ (e^+(\vec{p}) \gamma_0)^+$$

$$= \frac{g}{\sqrt{2}} \varepsilon_\mu^* \nu^+(\vec{q}) L \gamma^0 \gamma_\mu e(\vec{p})$$

we used $\gamma_\mu^+ = \gamma^0 \gamma_\mu \gamma^0$

$\gamma^{0+} = \gamma^0$

$(\gamma^0)^2 = \mathbb{1}$

$$L\gamma^0 = \left(\frac{1+\gamma_5}{2}\right)\gamma^0 = \gamma^0\left(\frac{1-\gamma_5}{2}\right) = \gamma^0 R.$$

$$\Rightarrow \mathcal{M}^+ = \frac{g}{\sqrt{2}} \varepsilon_\mu^* \bar{\nu}(\vec{q}) R \gamma_\mu e(\vec{p}) \quad (**)$$

$(*)$ & $(**)$ $\Rightarrow$

$$|\mathcal{M}|^2 = \sum_{spins} \mathcal{M}^+ \mathcal{M} = \frac{g^2}{2} \varepsilon_\mu^* \varepsilon_\nu \sum_{spins} \left(\bar{\nu}(\vec{q}) R \gamma_\mu e(\vec{p})\right) \times \left(\bar{e}(\vec{p}) \gamma_\nu L \nu(\vec{q})\right)$$

$$= \frac{g^2}{2} \varepsilon_\mu^* \varepsilon_\nu \, Tr(\not{q} R \gamma_\mu \not{p} \gamma_\nu L \not{q})$$

$$= \frac{g^2}{2} \varepsilon_\mu^* \varepsilon_\nu q_\alpha p_\beta \, Tr[\gamma_\alpha R \gamma_\mu \gamma_\beta \gamma_\nu L]$$

$$= \frac{g^2}{2} \varepsilon_\mu^* \varepsilon_\nu q_\alpha p_\beta \, Tr[\gamma_\alpha \left(\frac{1-\gamma_5}{2}\right) \gamma_\mu \gamma_\beta \gamma_\nu \left(\frac{1+\gamma_5}{2}\right)]$$

$$= g^2 \varepsilon_\mu^* \varepsilon_\mu q_\alpha p_\beta \times$$
$$\times [g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\beta} g_{\mu\nu} + g_{\alpha\nu} g_{\mu\beta} + i\varepsilon_{\alpha\mu\beta\nu}]$$

$$\boxed{= g^2 [(q \cdot \varepsilon^*)(p \cdot \varepsilon) - (\varepsilon^* \cdot \varepsilon)(p \cdot q) + (q \cdot \varepsilon)(p \cdot \varepsilon^*) + i\varepsilon_{\alpha\mu\beta\nu} q_\alpha \varepsilon_\mu^* p_\beta \varepsilon_\nu]} \quad (***)$$

i) $\varepsilon_+^\mu (+) = \frac{1}{\sqrt{2}} (0, 1, i, 0)$

$$p_\mu = \frac{m_W}{2} (1, \sin\theta, 0, \cos\theta)$$

$$q_W = \frac{m_W}{2} (1, -\sin\theta, 0, -\cos\theta).$$

$$\hookrightarrow |\mathcal{M}(+)|^2 = g^2 \frac{m_W^2}{4} (1 - \cos\theta)^2.$$

From the above, we find:

$$\frac{d\Gamma}{d\Omega} = \frac{1}{64\pi^2 m_W} |\mathcal{M}(+)|^2$$

$$= \frac{g^2 m_W}{64\pi^2} \times \frac{1}{4} (1 - \cos\theta)^2$$

$$\hookrightarrow \Gamma(+) = \int d\Omega \frac{d\Gamma(+)}{d\Omega} = \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \frac{d\Gamma}{d\Omega} = \frac{g^2 m_W}{48\pi}$$

b) $\varepsilon_T^\mu = \frac{1}{\sqrt{2}}(0; 1, -i, 0)$

From (***)

$$\hookrightarrow |\mathcal{M}(-)|^2 = \frac{g^2 m_W^2}{4}(1+\cos\theta)^2$$

$$\boxed{\Gamma(-) = \Gamma(+)}$$ → This is to be expected as the decay rate cannot depend on how we chose the axis.

c) $\varepsilon_L^\mu(0) = (0; 0, 0, 1)$

From (***)

$$\hookrightarrow |\mathcal{M}(0)|^2 = \frac{g^2 m_W^2}{2} \sin^2\theta.$$

$$\frac{d\Gamma}{d\Omega} = \frac{g^2 m_W^2}{64\pi^2} \times \frac{1}{2}\sin^2\theta$$

$$\boxed{\Gamma(0) = \Gamma(+) = \Gamma(-)}$$

d) Unpolirized W:

Average of all possible polorizations

$$|\mathcal{M}|^2 = \frac{1}{3}\left(|\mathcal{M}(+)|^2 + |\mathcal{M}(-)|^2 + |\mathcal{M}(0)|^2\right)$$

$$= \frac{g^2 m_W^2}{3}\left(\frac{1}{4}(1-\cos\theta)^2 + \frac{1}{4}(1+\sin\theta)^2 + \frac{1}{2}\sin^2\theta\right)$$

$$= \frac{g^2 m_W^2}{3}$$

After integrating:

$$\hookrightarrow \boxed{\Gamma = \Gamma(+) = \Gamma(-) = \Gamma(0)}$$

(e) Leptonic channel:

$$\Gamma(W \to e\,\nu_e) \approx \Gamma(W \to \mu\,\nu_\mu) \approx \Gamma(W \to \tau\,\nu_\tau)$$

$$\Gamma(W \to \text{leptons}) \simeq 3\Gamma(W \to e\nu_e)$$

Hadronic channel:

$$\Gamma(W \to \bar{q}q') \simeq 6\Gamma(W \to e\nu_e)$$

W-boson cannot decay to top quark

Therefore.

$$\Gamma(W \to \text{S.M.}) \simeq 9\Gamma(W \to e\nu_e)$$

$$\simeq \frac{g^2}{4\pi} \cdot \frac{3m_W}{4} \simeq 2\ \text{GeV}.$$