Spontaneous Symmetry Breaking of SO(3)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi^i)(\partial^\mu \varphi^i) - V(\varphi), \quad V(\varphi) = -\frac{\mu^2}{2}\varphi^i\varphi^i + \frac{\lambda}{4}(\varphi^i\varphi^i)^2, \quad i=1,2,3. \quad \mu^2, \lambda > 0$$

(a) Intuition: the fields φ^i enter the Lagrangian only as the norm („length") squared $(\varphi^i\varphi^i)$ $\Rightarrow$ we expect invariance under global SO(3) rotations of the fields.

Consider a rotation of the fields φ^i by an infinitesimal angle α: $\varphi^i \to \varphi^{i\prime} = \varphi^i + \alpha\, \varepsilon^{ijk} n^j \varphi^k$, i.e. $\delta\varphi^i = \varepsilon^{ijk} n^j \varphi^k$ where n^i is a constant unit vector.

Then $(\varphi^i\varphi^i)$ transforms as:

$$\varphi^i\varphi^i \to \varphi^{i\prime}\varphi^{i\prime} = (\varphi^i + \alpha\,\varepsilon^{ijk} n^j\varphi^k)(\varphi^i + \alpha\,\varepsilon^{ilm} n^l \varphi^m) + \mathcal{O}(\alpha^2) =$$

$$= \varphi^i\varphi^i + \alpha\,\varepsilon^{ijk} n^j \varphi^k\varphi^i + \underbrace{\alpha\,\varepsilon^{ilm} n^l \varphi^m \varphi^i}_{\text{re-label } l\to j,\ m\to k} + \mathcal{O}(\alpha^2) =$$

$$= \varphi^i\varphi^i + 2\alpha \underbrace{\varepsilon^{ijk}}_{\substack{\text{anti-symm. under}\\ \text{exchange } i\leftrightarrow k}} n^j \underbrace{\varphi^k\varphi^i}_{\substack{\text{symm. under}\\ \text{exchange } i\leftrightarrow k}} + \mathcal{O}(\alpha^2) = \varphi^i\varphi^i + \mathcal{O}(\alpha^2) \ \checkmark$$

Similarly one can show that also $(\partial_\mu\varphi^i)(\partial^\mu\varphi^i)$ is invariant under SO(3) rotations.

(b) The ground state is a homogeneous field in space-time, minimizing the potential $V(\varphi)$.

We can find the extrema of the potential by computing

$$0 \overset{!}{=} \frac{\partial V}{\partial \varphi^i} = -\mu^2\varphi^i + \lambda(\varphi^j\varphi^j)\varphi^i = \varphi^i\left(-\mu^2 + \lambda(\varphi^j\varphi^j)\right)$$

local maximum: $\varphi^i = 0$

global minima: $\boxed{(\varphi^i\varphi^i) = \varphi_0^2 \ ; \ \varphi_0 \equiv \frac{\mu}{\sqrt{\lambda}}}$

The vacuum manifold, i.e. the set of all possible ground states is a <u>two-dimensional sphere of radius φ_0.</u>

(c) As the ground state we can choose any point on the sphere; let us choose

$\varphi^1 = \varphi^2 = 0$

$\varphi^3 = \varphi_0$

The vacuum vector $\vec{\varphi}^{(0)} = (0, 0, \varphi_0)$ does not break the symmetry completely. There is a non-trivial subgroup of the group $SO(3)$, under which the vacuum vector is invariant: $\omega \vec{\varphi}^{(0)} = \vec{\varphi}^{(0)}$.

This subgroup is the group $SO(2)$ of rotations in the space of the fields about the third axis:

$\varphi^1 \to \varphi^1 \cos\alpha - \varphi^2 \sin\alpha$

$\varphi^2 \to \varphi^1 \sin\alpha + \varphi^2 \cos\alpha$

$\varphi^3 \to \varphi^3$

Reminder: Spontaneous Symmetry Breaking (SSB):

The Lagrangian obeys a symmetry, but the vacuum does not.

Note: in case of gauge symm., the gauge redundancy is still there! (see PS 2).

Goldstone's Theorem
For every spontaneously broken continuous symmetry, the theory must contain a massless particle.

The massless fields that arise through SSB are called Goldstone bosons.

Of the three generators of the group $SO(3)$, one generator annihilates the vacuum $\vec{\varphi}^{(0)} = (0, 0, \varphi_0)$: $T_3 \vec{\varphi}^{(0)} = 0$.

This is the generator of the unbroken subgroup $SO(2)$: for ω close to unity, from the above, we have for $\varepsilon \ll 1$:

$\omega \vec{\varphi}^{(0)} = (1 + \varepsilon T_3) \vec{\varphi}^{(0)} = \vec{\varphi}^{(0)} \iff T_3 \vec{\varphi}^{(0)} = 0.$

The two other generators (and any linear combinations of them) do not annihilate the vacuum, otherwise the unbroken subgroup should be larger than $SO(2)$.

$$\left.\begin{array}{l} SO(3):\ 3 \text{ generators}: (T_a)_{bc} = -\varepsilon_{abc} \\ SO(2):\ 1 \text{ generator}: T_3 \text{ in our case} \end{array}\right\} \Rightarrow \begin{array}{l} 2 \text{ broken generators } T_1, T_2; \\ \text{i.e. they do not annihilate} \\ \text{the vacuum } \vec{\varphi}^{(0)}. \end{array}$$

$\Rightarrow$ We expect 2 Nambu-Goldstone bosons.

Let us now compute the quadratic part of the Lagrangian for the perturbations on top of the vacuum, and in particular, determine which perturbations are massless Nambu-Goldstone modes:

Let us introduce the fields of perturbations χ, θ^1, θ^2 such that

$\varphi^1 = \theta^1$

$\varphi^2 = \theta^2$

$\varphi^3 = \varphi_0 + \chi$

The potential term in the Lagrangian for the perturbations has the form

$$V = -\frac{\mu^2}{2}\Big[(\theta^1)^2 + (\theta^2)^2 + \underbrace{(\varphi_0+\chi)^2}_{=\varphi_0^2 + \cancel{2\varphi_0\chi} + \chi^2}\Big] + \frac{\lambda}{4}\underbrace{\Big[(\theta^1)^2 + (\theta^2)^2 + (\varphi_0+\chi)^2\Big]^2}_{=\frac{\lambda}{2}\varphi_0^2[(\theta^1)^2+(\theta^2)^2] + \frac{3}{2}\lambda\varphi_0^2\chi^2 + \cancel{\lambda\varphi_0^3\chi} + \text{const} + \text{h.o.t.}}$$

and the kinetic term is equal to

$$\mathcal{L}_{kin} = \frac{1}{2}(\partial_\mu\theta^1)^2 + \frac{1}{2}(\partial_\mu\theta^2)^2 + \frac{1}{2}\Big(\underbrace{\partial_\mu[\overbrace{\varphi_0}^{=\text{const}} + \chi]}_{=\partial_\mu\chi}\Big)^2$$

$\Rightarrow \mathcal{L} = \mathcal{L}_{kin} - V$ is invariant under $SO(2)$ rotations of θ^1, θ^2 since it contains only combinations of type $(\theta^1)^2 + (\theta^2)^2$. Of course, it does not have the full $SO(3)$ symmetry.

Quadratic part of the potential:

$$V^{(2)} = \underbrace{\Big(-\frac{\mu^2}{2} + \underbrace{\frac{\lambda}{2}\varphi_0^2}_{=\frac{\mu^2}{2}}\Big)}_{=0}\Big[(\theta^1)^2 + (\theta^2)^2\Big] + \Big(-\frac{\mu^2}{2} + \underbrace{\frac{3}{2}\lambda\varphi_0^2}_{=\frac{3}{2}\mu^2}\Big)\chi^2 = \mu^2\chi^2$$

Quadratic part of the Lagrangian for the perturbations is

$$\boxed{\mathcal{L}^{(2)} = \frac{1}{2}(\partial_\mu\theta^1)^2 + \frac{1}{2}(\partial_\mu\theta^2)^2 + \frac{1}{2}(\partial_\mu\chi)^2 - \mu^2\chi^2}$$

$\Rightarrow \theta^1, \theta^2$ are the 2 massless Nambu-Goldstone fields.

(d) Let us now define $\vec{\varphi} = (\varphi^1, \varphi^2, \varphi^3)^T$ and gauge the theory by promoting $SO(3)$ to a local symmetry:
Then $\vec{\varphi} \to \vec{\varphi}' = U\vec{\varphi}$ w/ $U = U(x) = \exp(i\alpha(x)) = \exp(i\alpha^a(x) t^a)$
where $t^a_{ij} = iT^a_{ij}$, and $A_\mu \to A'_\mu = U\left[A_\mu + \frac{1}{g}\partial_\mu \alpha\right]U^{-1}$ with $A_\mu = A^a_\mu t^a$.
The partial derivative ∂_μ is replaced by $D_\mu = \partial_\mu - igA_\mu = \partial_\mu + gA^a_\mu T^a$.
Similarly to PS3: $D_\mu \vec{\varphi} \to D'_\mu \vec{\varphi}' = U D_\mu \vec{\varphi}$.
We now have a locally gauge invariant lagrangian:
$\mathcal{L}_{loc} = \frac{1}{2}(D_\mu \vec{\varphi})^T(D^\mu \vec{\varphi}) - V(\varphi)$, which in addition to the globally invariant lagrangian $\mathcal{L}_{glob} = \frac{1}{2}(\partial_\mu \vec{\varphi})^T(\partial^\mu \vec{\varphi}) - V(\varphi)$ we had before, now also contains the interaction lagrangian

$$\mathcal{L}_{int} = \frac{1}{2}g(\partial_\mu \vec{\varphi})^T(A^{\mu a}T^a \vec{\varphi}) + \frac{1}{2}g(A^a_\mu T^a \vec{\varphi})^T(\partial^\mu \vec{\varphi}) + \frac{g^2}{2}(A^a_\mu T^a \vec{\varphi})^T(A^{\mu b}T^b \vec{\varphi}) =$$

$$= g A^a_\mu (\partial^\mu \varphi_i) T^a_{ij} \varphi_j + \frac{g^2}{2} A^a_\mu A^{\mu b} (T^a \varphi)_i (T^b \varphi)_i$$

To make this interaction physical, A^a_μ need to propagate.
So we add a kinetic term for the gauge fields
$\mathcal{L}_{gf} = -\frac{1}{2}\mathrm{tr}\left[F_{\mu\nu}F^{\mu\nu}\right]$ w/ $F_{\mu\nu} = F^a_{\mu\nu} t^a$ and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$.
The complete lagrangian is then given by:

$$\boxed{\mathcal{L} = \mathcal{L}_{loc} + \mathcal{L}_{gf} = \mathcal{L}_{glob} + \mathcal{L}_{int} + \mathcal{L}_{gf} = \frac{1}{2}(D_\mu \vec{\varphi})^T(D^\mu \vec{\varphi}) - V(\varphi) - \frac{1}{2}\mathrm{tr}\left[F_{\mu\nu}F^{\mu\nu}\right]}$$

(e) To find the mass spectrum of the gauge fields A^a_μ, we expand $\frac{g^2}{2}(A^a_\mu T^a \vec{\varphi})^T(A^{\mu b}T^b \vec{\varphi}) \subset \mathcal{L}_{int}$ around the vacuum $\vec{\varphi}^{(0)} = (0, 0, \varphi_0)$:

$$\frac{g^2}{2}(A^a_\mu T^a \vec{\varphi})^T(A^{\mu b}T^b \vec{\varphi}) = \frac{g^2 \varphi_0^2}{2}\left[(A^1_\mu)^2 + (A^2_\mu)^2\right] = \frac{1}{2}M_{ab} A^a_\mu A^{\mu b}$$

$$\Rightarrow \boxed{\begin{array}{l} m_{A^1} = m_{A^2} = g\varphi_0 \\ m_{A^3} = 0 \end{array}}$$

That is, as expected, 2 gauge bosons corresponding to the 2 broken generators each acquired a mass. Also, since $T_3 \vec{\varphi}^{(0)} = 0$, as expected, $m_{A^3} = 0$.