

“QCD AND STANDARD MODEL”
Problem Set 4

1. Fermions as representations of the Lorentz group

We start from the left and right two-components Weyl spinors

$$u_L^\alpha(x) = \begin{pmatrix} u_{L1}(x) \\ u_{L2}(x) \end{pmatrix} \in \tau_{\frac{1}{2}0}, \quad u_{R\dot{\alpha}}(x) = \begin{pmatrix} u_{R1}(x) \\ u_{R2}(x) \end{pmatrix} \in \tau_{0\frac{1}{2}}, \quad (1)$$

where $\tau_{j^+j^-}$ denotes the representation of $SO(3,1)$ and j^+ , j^- label the representations of the $SU(2)$ sub-algebras. For the sake of simplicity, we drop the spinorial indices α and $\dot{\alpha}$ in what follows. Under Lorentz transformations, the spinors behave as

$$u'_{L,R}(x') = S_{L,R} u_{L,R}(x), \quad (2)$$

with the $SL(2, \mathbb{C})$ matrices

$$S_{L,R} = e^{-\frac{i\sigma_j}{2}(\theta_j \mp i\phi_j)}. \quad (3a)$$

Here, σ_j ($j = 1, 2, 3$) are the Pauli matrices, while θ_j, ϕ_j are the angle and rapidity parameters of the Lorentz group, respectively.

- a) Prove the following properties of the $SL(2, \mathbb{C})$ matrices :

$$S_L^{-1} = S_R^\dagger, \quad (4a)$$

$$\sigma_2 S_L \sigma_2 = S_R^* \quad (4b)$$

$$S_L^T = \sigma_2 S_L^{-1} \sigma_2. \quad (4c)$$

Of course, there are 3 similar identities that one gets by swapping $L \leftrightarrow R$.

- b) Use the relations (4) to prove that
 — Any left-handed Weyl spinor u_L is such that $\sigma_2 u_L^* \in \tau_{0\frac{1}{2}}$;
 — Adding another left-handed spinor v_L , we have $v_L^T \sigma_2 u_L \in \tau_{00}$, *i.e.*, it is a scalar;
 — $u_L^\dagger(x) \sigma_-^\mu u_L(x) \in \tau_{\frac{1}{2}\frac{1}{2}}$, where $\sigma_-^\mu = (I, -\sigma^j)$ and I is the 2×2 identity matrix.

How can we use these properties to guess the Lagrangian density $\mathcal{L}[u_L, u_R]$?

- c) However $SL(2, \mathbb{C})$ matrices realise a double covering of the full Lorentz group and the two chiralities mix under parity transformations. One has to introduce the *Dirac bispinor*

$$\psi(x) = \begin{pmatrix} u_L(x) \\ u_R(x) \end{pmatrix}, \quad (5)$$

and the Dirac matrices in the so-called *Weyl* or *chiral basis*

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_+^\mu \\ \sigma_-^\mu & 0 \end{pmatrix}, \quad \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (6)$$

with $\sigma_\pm^\mu = (I, \sigma^j)$. Prove that the transformation law for ψ reads

$$\psi'(x') = e^{i\theta_{\mu\nu}\Sigma^{\mu\nu}} \psi(x), \quad (7)$$

if one introduces the spin tensor

$$\Sigma^{\mu\nu} = \frac{1}{4i}[\gamma^\mu, \gamma^\nu] . \quad (8)$$

Hint : You may find useful to employ the transformations (3) together with the definition (5), and to compute the total variation $\Delta\psi(x) = \psi'(x') - \psi(x)$ by considering relativistic boosts and rotations separately.

- d) Show that the Dirac matrices do realise a matrix representation of the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} , \quad \{\gamma^\mu, \gamma_5\} = 0 . \quad (9)$$

Use it to prove the properties

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0 , \quad \gamma_5^\dagger = \gamma_5 . \quad (10)$$

- e) By using the relations of the previous points and the adjoint spinor $\bar{\psi} = \psi^\dagger \gamma^0$, show that $\bar{\psi}\psi$ is a scalar, whereas $\bar{\psi}\gamma^\mu\psi$ transforms as a vector. Therefore, we can construct the renowned Dirac Lagrangian

$$\mathcal{L}_D[\psi] = i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\bar{\psi}(x)\psi(x) \quad (11)$$

for a free bispinor of mass m .

- f) Prove that one can construct two projectors

$$L = \frac{1}{2}(1 + \gamma_5) , \quad R = \frac{1}{2}(1 - \gamma_5) , \quad (12)$$

such that $L\psi = (u_L, 0)^T$ and $R\psi = (0, u_R)^T$.