Non-Abelian gauge theories

a) $\mathcal{L}_{YM} = -\frac{1}{2} Tr(F_{\mu\nu} F^{\mu\nu})$

$= -\frac{1}{2} Tr(F_{\mu\nu}^a T^a F^{\mu\nu,b} T^b)$

$= -\frac{1}{2} F_{\mu\nu}^a F^{\mu\nu,b} \underbrace{Tr(T^a T^b)}_{= \frac{1}{2}\delta^{ab}}$

$= -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a}$

Normalization condition we used for SU(2) & SU(3) (Set 1). Can of course be used for SU(N) in general.

b) $\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_D$ invariant under non-Abelian gauge transformation

$$\boxed{\begin{aligned} A_\mu(x) &\longrightarrow A_\mu'(x) = U(x)\left[A(x) + \tfrac{1}{g}\partial_\mu \Theta(x)\right]U^{-1}(x) \\ \Psi(x) &\longrightarrow \Psi'(x) = U(x)\,\Psi(x) \end{aligned}}$$

where we used $U(x) = \exp(i\Theta^a(x) T^a) \equiv \exp(i\Theta(x))$.

$\hookrightarrow m\bar{\Psi}\Psi \longrightarrow m\bar{\Psi}\underbrace{U^\dagger U}_{=\mathbb{1}}\Psi = m\bar{\Psi}\Psi$ ✓

$\hookrightarrow D_\mu\Psi \longrightarrow D_\mu'\Psi' = (\partial_\mu - igA_\mu')\Psi'$

$= (\partial_\mu - igU[A + \frac{1}{g}\partial\Theta]U^{-1})(U\Psi)$

$= \underbrace{\partial_\mu U\Psi}_{= U i \partial_\mu \Theta \Psi} + U\partial\Psi - igU[A + \frac{1}{g}\partial\Theta]\Psi$

$= U(\partial_\mu - igA)\Psi = U D_\mu \Psi$

$\Longrightarrow i\bar{\Psi}\not{D}\Psi \longrightarrow i\bar{\Psi}'\not{D}'\Psi' = i\bar{\Psi}\underbrace{U^\dagger U}_{=\mathbb{1}}\not{D}\Psi$

$= i\bar{\Psi}\not{D}\Psi$ ✓

↳ Let's rewrite transformation of cov. derivative

$$D_\mu' \Psi' = U D_\mu \Psi = U D_\mu U^{-1} U \Psi \quad \forall \Psi$$

$$\Longrightarrow \boxed{D_\mu' = U D_\mu U^{-1}}$$

Use result from e)

$$-ig F_{\mu\nu} \Psi \rightarrow -ig F_{\mu\nu}' \Psi' = [D_\mu', D_\nu'] \Psi'$$
$$= (D_\mu' D_\nu' - D_\nu' D_\mu') \Psi'$$
$$= (U D_\mu \underbrace{U^{-1} U}_{=1} D_\nu U^{-1} - U D_\nu \underbrace{U^{-1} U}_{=1} D_\mu U^{-1}) \Psi'$$
$$= U [D_\mu, D_\nu] U^{-1} \Psi' \quad \forall \Psi'$$

$$\Longrightarrow \boxed{F_{\mu\nu}' = U F_{\mu\nu} U^{-1}}$$

$$\Longrightarrow Tr(F_{\mu\nu} F^{\mu\nu}) \rightarrow Tr(U F_{\mu\nu} \underbrace{U^{-1} U}_{=1} F^{\mu\nu} U^{-1})$$
$$= Tr(\underbrace{U^{-1} U}_{=1} F_{\mu\nu} F^{\mu\nu})$$
$$= Tr(F_{\mu\nu} F^{\mu\nu}) \quad \checkmark$$

c) For global $SU(N)$ trafo:

$$\Psi' = (1 + i\theta^a T^a)\Psi = \Psi + i\theta^a T^a \Psi$$
$$A_\mu' = (1 + i\theta^a T^a) A_\mu (1 - i\theta^b T^b) = A_\mu + i\theta^a (T^a A_\mu - A_\mu T^a)$$
$$= A_\mu + i\theta^a A_\mu^b \underbrace{[T^a, T^b]}_{= i f^{abc} T^c} = A_\mu - \theta^a A_\mu^b T^c f^{abc}$$
$$= A_\mu^c T^c - \theta^a A_\mu^b T^c f^{abc} = (A_\mu^c - \theta^a A_\mu^b f^{abc}) T^c$$

$$\Longrightarrow \boxed{\delta^a \Psi = i T^a \Psi} \quad \delta^a \ldots \text{ variation in } \theta^a\text{-direction}$$
$$\boxed{\delta^a A_\mu^c = -A_\mu^b f^{abc}}$$

Use usual equations for Noether current but take into account that for every θ^a there will be a Noether current: "$J_\mu \rightarrow J_\mu^a$"

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$$\boxed{J^{\mu,a} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} \delta^a \Phi} \quad \text{for any kind of field } \Phi$$

$$\Longrightarrow J_\mu{}^a = \frac{\partial \mathcal{L}}{\partial \partial^\mu \Psi} \delta^a \Psi + \frac{\partial \mathcal{L}}{\partial \partial^\mu A^{\nu,c}} \delta^a A^{\nu,c}$$

$$\equiv (J_{matter})_\mu{}^a + (J_{YM})_\mu{}^a$$

$$\hookrightarrow (J_{matter})_\mu{}^a = (i\bar{\Psi}\gamma_\mu)(iT^a\Psi) = -\bar{\Psi}\gamma_\mu T^a \Psi$$

$$\hookrightarrow (J_{YM})_\mu{}^a = \frac{\partial \mathcal{L}}{\partial F_{\alpha\beta}{}^{\tilde{a}}} \frac{\partial F_{\alpha\beta}{}^{\tilde{a}}}{\partial \partial^\mu A^{\nu,c}} \delta^a A^{\nu,c}$$

$$= -\frac{1}{4}(2F^{\alpha\beta,\tilde{a}}) \frac{\partial(\partial_\alpha A_\beta^{\tilde{a}} - \partial_\beta A_\alpha^{\tilde{a}} + g f^{\tilde{a}\tilde{b}\tilde{c}} A_\alpha^{\tilde{b}} A_\beta^{\tilde{c}})}{\partial \partial^\mu A^{\nu,c}} \delta^a A^{\nu,c}$$

$$= -\frac{1}{2} F^{\alpha\beta,\tilde{a}} (\delta^{\tilde{a}c} \eta_{\alpha\mu}\eta_{\beta\nu} - \delta^{\tilde{a}c} \eta_{\alpha\nu}\eta_{\beta\mu}) \delta^a A^{\nu,c}$$

$$= -\frac{1}{2}(F_{\mu\nu}{}^c \underbrace{- F_{\nu\mu}{}^c}_{= + F_{\mu\nu}{}^c}) \delta^a A^{\nu c}$$

$$= -F_{\mu\nu}{}^c \delta^a A^{\nu c}$$

$$= + F_{\mu\nu}{}^c A^{\nu,b} f^{abc}$$

$$\Longrightarrow \boxed{J_\mu{}^a = -\bar{\Psi}\gamma_\mu T^a \Psi + F_{\mu\nu}{}^c A^{\nu,b} f^{abc}}$$

This current requires some discussion (see Schwartz book for more details)

- Conserved: $\partial^\mu J_\mu{}^a = 0$ (per definition of a Noether current)
- Not gauge invariant, not even covariant. $\Longrightarrow$ unphysical
- Consider only $(J_{matter})_\mu$, which is gauge covariant. However, this current is only covariantly conserved, i.e. $D^\mu (J_{matter})_\mu{}^a = 0$

From this equal statements follow for the charge

$$Q^a = \int d^3x \, J_0^a$$

- Conserved, $\partial_t Q^a = 0$ (again per definition of a Noether charge)
- Not gauge invariant, not even covariant $\longrightarrow$ unphysical
- Consider only $Q^a_{matter} = \int d^3x \, (J_{matter})^a_0$, which is gauge covariant. However, this charge is not conserved.

All in all we can say that one needs to be very careful w/ the analogy to an Abelian theory like QED. In QED we had Gauss law following from inserting the eom into the definition of Q. This is not the case in non-Abelian theories, so the gauge fields are bound up w/ the matter fields in an intricate and non-linear way:

$$\text{QED: } \quad Q = \int_\Omega d^3x \, J_0 \overset{eom}{=} \int_\Omega d^3x \, \partial_i \underbrace{F^{i0}}_{=E^i} = \int_{\partial\Omega} d\underline{S} \cdot \underline{E} \qquad \text{Gauß-law}$$

$$\text{YM: } \quad Q^a = \int_\Omega d^3x \, J_0^a \overset{eom}{=} \int_\Omega d^3x \left(\partial_i F^{i0,a} + \underbrace{g f^{abc} A_i^b F^{i0,c}}_{\text{spoils Gauß-law}} \right)$$

d) eom($\bar{\Psi}$): $\boxed{(i\not{D} - m)\bar{\Psi} = 0 \iff (i\not{\partial} - m)\bar{\Psi} = -g \not{A} \bar{\Psi}}$

eom(A): $\boxed{\partial_\mu F^{\mu\nu,a} + g f^{abc} A_\mu^b F^{\mu\nu,c} = J^{\nu,a}_{matter} = -g \bar{\Psi} \gamma^\nu T^a \Psi}$

This term was not present in QED. Since it is ~~quadratic~~ of higher power in A, the whole eom is non-linear and the gauge bosons have selfinteraction.

e) $[D_\mu, D_\nu]\Psi = [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu]\Psi$

$$= (\partial_\mu - igA_\mu)[(\partial_\nu - igA_\nu)\Psi] - (\partial_\nu - igA_\nu)[(\partial_\mu - igA_\mu)\Psi]$$

$$= \cancel{\partial_\mu \partial_\nu \Psi} - ig\,\partial_\mu A_\nu \Psi \cancel{- igA_\nu \partial_\mu \Psi} - igA_\mu(\partial_\nu - igA_\nu)\Psi$$
$$\cancel{-\partial_\nu \partial_\mu \Psi} + ig\,\partial_\nu A_\mu \Psi \cancel{+ igA_\mu \partial_\nu \Psi} + igA_\nu(\partial_\mu - igA_\mu)\Psi$$

$$= -ig\,\partial_\mu A_\nu \Psi - g^2 A_\mu A_\nu \Psi + ig\,\partial_\nu A_\mu \Psi + g^2 A_\nu A_\mu \Psi$$

$$= -ig\left[\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\right]\Psi$$

$$\boxed{[D_\mu, D_\nu]\Psi = -ig\, F_{\mu\nu}\, \Psi}$$

Unfortunately, we can not appreciate this relation without a longer excursion in diferential geometry.

f) $D_\rho F_{\mu\nu} + D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu}$

$$= \frac{1}{-ig}\left(D_\rho [D_\mu, D_\nu] + D_\mu [D_\nu, D_\rho] + D_\nu [D_\rho, D_\mu]\right)$$

$$= \frac{1}{-ig}\left(\underline{D_\rho D_\mu D_\nu} - \underline{D_\rho D_\nu D_\mu} + \underline{D_\mu D_\nu D_\rho} - \underline{D_\mu D_\rho D_\nu} + \underline{D_\nu D_\rho D_\mu} - \underline{D_\nu D_\mu D_\rho}\right.$$

$$= \frac{1}{-ig}\left([D_\mu, D_\nu]D_\rho + [D_\rho, D_\mu]D_\nu + [D_\nu, D_\rho]D_\mu\right)$$

$$\Longrightarrow \boxed{[D_\rho, [D_\mu, D_\nu]] + [D_\mu, [D_\nu, D_\rho]] + [D_\nu, [D_\rho, D_\mu]] = 0} \quad \text{Jacobi-Identity}$$

$$[D_\mu, [D_\nu, D_\rho]]\Psi = [D_\mu, ig F_{\nu\rho}]\Psi$$
$$= ig[D_\mu (F_{\nu\rho}\Psi) - F_{\nu\rho} D_\mu \Psi]$$
$$= ig[D_\mu F_{\nu\rho}\Psi + \cancel{F_{\nu\rho} D_\mu \Psi} - \cancel{F_{\nu\rho} D_\mu \Psi}]$$

Putting this into the Jacobi identity we find the Bianchi identity.

g) $\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ Intuition for Abelian case:

$$F = \begin{pmatrix} 0 & -\underline{E} \\ \underline{E} & \varepsilon \underline{B} \end{pmatrix}$$
$$\tilde{F} = \begin{pmatrix} 0 & -\underline{B} \\ \underline{B} & \varepsilon \underline{E} \end{pmatrix}$$

<u>Abelian case:</u>

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2} F_{\mu\nu}\, \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$
$$= \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha)$$

rename indices and use anti-sym of ε.
$$= \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}(2\partial_\mu A_\nu)(2\partial_\alpha A_\beta)$$
$$= 2\,\varepsilon^{\mu\nu\alpha\beta}\,\partial_\mu A_\nu \partial_\alpha A_\beta$$

Partial int.
$$= \underbrace{-2\,\varepsilon^{\mu\nu\alpha\beta} A_\nu \partial_\mu \partial_\alpha A_\beta}_{=0} + 2\,\varepsilon^{\mu\nu\alpha\beta}\partial_\mu (A_\nu \partial_\alpha A_\beta)$$

$$= \partial_\mu \big(\underbrace{2\,\varepsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta}_{\equiv K^\mu}\big)$$

$\therefore$ total derivative! Does not contribute to eom since variation vanishes at boundary

YM: $\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) = \frac{1}{2} F_{\mu\nu}{}^{a} \tilde{F}^{\mu\nu,a}$

$$= \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c\right)\left(\partial_\alpha A_\beta^a - \partial_\beta A_\alpha^a + g f^{a\tilde{b}\tilde{c}} A_\alpha^{\tilde{b}} A_\beta^{\tilde{c}}\right)$$

$$= \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\left(2\partial_\mu A_\nu^a + g f^{abc} A_\mu^b A_\nu^c\right)\left(2\partial_\alpha A_\beta^a + g f^{a\tilde{b}\tilde{c}} A_\alpha^{\tilde{b}} A_\beta^{\tilde{c}}\right)$$

$$= \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\Big(4\partial_\mu A_\nu^a \partial_\alpha A_\beta^a + 2\partial_\mu A_\nu^a g f^{a\tilde{b}\tilde{c}} A_\alpha^{\tilde{b}} A_\beta^{\tilde{c}} + 2\partial_\alpha A_\beta^a g f^{abc} A_\mu^b A_\nu^c + \underbrace{g^2 f^{abc} f^{a\tilde{b}\tilde{c}} A_\mu^b A_\nu^c A_\alpha^{\tilde{b}} A_\beta^{\tilde{c}}}\Big)$$

symmetric under $b \leftrightarrow \tilde{b}$, $c \leftrightarrow \tilde{c}$, but contracted with antisym ε-tensor. $= 0$

$$= \varepsilon^{\mu\nu\alpha\beta}\left(2\partial_\mu A_\nu^a \partial_\alpha A_\beta^a + g\,\partial_\mu A_\nu^a A_\alpha^b A_\beta^c f^{abc} + g\,\partial_\alpha A_\beta^a A_\mu^b A_\nu^c f^{abc}\right)$$

Rename indices in last term

$$= 2\varepsilon^{\mu\nu\alpha\beta}\left(\partial_\mu A_\nu^a \partial_\alpha A_\beta^a + g\,\partial_\mu A_\nu^a A_\alpha^b A_\beta^c f^{abc}\right)$$

$$= -\underbrace{A_\nu^a \partial_\mu\partial_\alpha A_\beta^a}_{=0,\ \text{sym.}} + \partial_\mu(A_\nu^a \partial_\alpha A_\beta^a)$$

$$\frac{g}{3}\left(3\partial_\mu A_\nu^a A_\alpha^b A_\beta^c f^{abc}\right)$$

$$= \frac{g}{3}\partial_\mu\left(A_\nu^a A_\alpha^b A_\beta^c f^{abc}\right)$$

where we work out all 3 terms and renamed the indices s.t. the derivative acts on each A_ν, A_α, A_β

$$\Rightarrow \boxed{K^\mu = 2\varepsilon^{\mu\nu\alpha\beta}\left(A_\nu^a \partial_\alpha A_\beta^a + \frac{g}{3} f^{abc} A_\nu^a A_\alpha^b A_\beta^c\right)}$$

Again this is a boundary term and thus does not contribute into the eom. Nevertheless this term is allowed in the $\mathcal{L}$.

Big spoiler for advanced QFT lecture of prof. Dvali: In fact, YM theories include field configurations called instantons that make this term physical!!! These field configurations are non-perturbative solutions to the equations of motions and are related to non-trivial topological properties of YM theories.

Remarks

- In f) we used the Leibniz rule (= product rule) of the covariant derivative

↳ First of all if a field $\Phi \in V$ lives in a Lie-algebra rep. $\rho: \mathfrak{g} \to \mathrm{End}(V)$, then it is implicitly understood that $D_\mu = \partial_\mu - igA_\mu$ should be seen as

$$D_\mu \Phi = \partial_\mu \Phi - ig \underbrace{\rho(A_\mu)(\Phi)}_{\text{rep of } A_\mu \text{ acting on } \Phi}$$

E.g.: $\rho_{\text{trivial}}(A_\mu)(\Phi) = 0$

$\rho_{\text{fund}}(A_\mu)(\Phi) = A_\mu \Phi$

$\rho_{\text{adjoint}}(A_\mu)(\Phi) = [A_\mu, \Phi] = A_\mu^a \Phi^b f^{abc} T^c$

$\rho_{\substack{\text{any d-dim}\\ \text{matrix rep}}}(A_\mu)(\Phi) = [A_\mu, \Phi]$

↳ Consider now $\phi = \phi_1 \otimes \phi_2$ w/ ϕ_1 & ϕ_2 in some reps. Then by the "product rule" of the commutator

$$\rho(A_\mu)(\phi) = [A_\mu, \phi_1 \otimes \phi_2] = [A_\mu, \phi_1] \otimes \phi_2 + \phi_1 \otimes [A_\mu, \phi_2]$$
$$= \rho_1(A_\mu)(\phi_1) \otimes \phi_2 + \phi_1 \otimes \rho_2(A_\mu)(\phi_2)$$

(Alternatively, write $A_\mu = A_\mu^a T^a$ and write out the commutator) $= A^a(T_1^a \otimes \mathbb{1} + \mathbb{1} \otimes T_2^a)$

- The Bianchi identity is understood with D_ρ acting on $F_{\mu\nu}$. But due to the Leibniz rule it also works if acting on some field.