

“QCD AND STANDARD MODEL”  
**Problem Set 12**

## Chiral symmetry breaking and sigma models

At the classical level the Lagrangian of QCD with two massless quarks (u and d) is invariant under the global symmetry group

$$SU(2)_L \otimes SU(2)_R \otimes U(1)_A \otimes U(1)_V \quad (1)$$

In this exercise we focus on the global chiral symmetries  $SU(2)_L$  and  $SU(2)_R$ .

- a) Write down the QCD Lagrangian for two massless quarks and identify the global chiral symmetries  $SU(2)_L$  and  $SU(2)_R$  under which it is invariant. How would this symmetry group generalize if we add more massless quarks to the theory?
- b) Argue that a condensate

$$\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = v^3 \quad (2)$$

breaks  $SU(2)_L \otimes SU(2)_R$  down to its diagonal subgroup  $SU(2)_V$ . How many Goldstone bosons emerge? How many Goldstone bosons emerge in the case of three massless quarks when

$$SU(3)_L \otimes SU(3)_R \longrightarrow SU(3)_V ? \quad (3)$$

- c) In particle physics, we often use phenomenological so-called "sigma model Lagrangians" in order to describe the dynamics of the (pseudo-)Goldstone bosons at low energies. We shall now investigate Gell-Mann and Levy linear and non-linear sigma models which correspond to the QCD Lagrangian with two massless quarks.

Consider the Lagrangian density of the linear sigma model with the four scalar fields  $\phi^{ij}$  transforming linearly under  $SU(2)_L \otimes SU(2)_R$ ,

$$\mathcal{L} = \frac{1}{2} |\partial_\mu \phi|^2 + \frac{\mu^2}{2} |\phi|^2 - \frac{\lambda}{4} |\phi|^4 \quad (4)$$

where  $\lambda > 0$  and barred indices belong to  $SU(2)_R$ . Minimize the potential, take the VEV  $v$  of the scalar fields in such a way that the symmetry is broken into the diagonal subgroup and identify the (pseudo-)Goldstone bosons  $\pi^1$ ,  $\pi^2$  and  $\pi^3$ . Expand the Lagrangian around the VEV by introducing a fluctuation,  $\sigma(x)$ , and write it down in terms of  $\sigma$  and  $\pi^a$  ( $a = 1, 2, 3$ ).

- d) Argue what happens physically to  $\sigma$  in the double scaling limit  $\mu^2 \rightarrow \infty$ ,  $\lambda \rightarrow \infty$ ,  $v^2$  fixed. Argue that the fields satisfy the constraint  $\pi^a \pi^a + \sigma^2 = -2v\sigma$  in this limit.
- e) Plugging this constraint into the Lagrangian of the linear sigma model leads to the so-called non-linear sigma model for the three Goldstone bosons  $\pi^a$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{2} \frac{\pi^a \partial_\mu \pi^a \pi^b \partial^\mu \pi^b}{v^2 - \pi^a \pi^a} \quad (5)$$

Expand this Lagrangian and show that it can be written as

$$\mathcal{L} = \frac{1}{2} \partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{6v^2} ((\pi^a \partial_\mu \pi^a)^2 - \pi^a \pi^a \partial_\mu \pi^b \partial^\mu \pi^b) + \mathcal{O}(\pi^6) \quad (6)$$

f) In the so-called exponential representation with an  $SU(2)$  field  $U$

$$U = \exp \left\{ i \frac{\pi^a \sigma^a}{f_\pi} \right\} , \quad (7)$$

with  $f_\pi \approx v$ , we can construct the effective Lagrangian for the non-linear sigma model

$$\mathcal{L} = \frac{v^2}{4} \text{Tr} (\partial_\mu U^\dagger \partial^\mu U) , \quad (8)$$

which is also called chiral Lagrangian. Convince yourself that this object is invariant under  $SU(2)$ . Under what representation of  $SU(2)$  do the fields  $\pi^a$  transform?

- g) Argue that we can organise the chiral Lagrangian in terms of the number of derivatives of the exponential  $U(x)$ . Why there are no terms containing only  $U(x)$  and without derivatives?
- h) Show that a pion mass term can be introduced in this language by adding

$$\delta\mathcal{L}_{\text{mass}} = v^3 \text{Tr} (MU + M^\dagger U^\dagger) \quad (9)$$

where

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} e^{-i\theta/2} . \quad (10)$$

Argue that we cannot remove the phase by making an  $SU(2)_L \times SU(2)_R$  transformation. However, we can trust the experimental value  $|\theta| < 10^{-9}$  and set it to zero. Guess how  $M$  should transform in such a way that  $\delta\mathcal{L}_{\text{mass}}$  is invariant under  $SU(2)_L \times SU(2)_R$ .