"QCD and Standard Model"
Problem Set 11

1. The Eightfold Way

In this exercise we shall study the Eightfold Way, which was an important step for discovering the quark model and QCD.

a) As a warm-up, consider SU(2) and derive explicitly the generators in the representation spin-$\frac{3}{2}$. *Hint : Start by constructing the ladder operators $J_\pm$. Remember that they act on the states of any irreducible representation (irrep) of SU(2) as*

$$J_\pm |j, m_j >= \sqrt{j(j+1) - m_j(m_j \pm 1)} |j, m_j \pm 1 > .$$

We can use the irrep's of SU(2) to describe particles that carry isospin. For example, the nucleon carries isospin $\frac{1}{2}$, meaning that its two states (proton and neutron) transform in the fundamental representation of SU(2). We write

$$|p >= |1/2, 1/2 >, \quad |n >= |1/2, -1/2 > .$$

Similarly, the pion carries isospin 1 and hence transforms in the 3-dimensional representation of SU(2).

b) Now consider the following processes of pion-nucleon scattering due to the strong interactions (ignore all other interactions)

$$\begin{aligned} A) &\quad \pi^+ + p \to \pi^+ + p, \\ B) &\quad \pi^0 + p \to \pi^0 + p, \\ C) &\quad \pi^+ + n \to \pi^0 + p, \\ D) &\quad \pi^0 + n \to \pi^- + p. \end{aligned}$$

Using only the fact that isospin is conserved in strong interactions, determine the ratio of the amplitudes. *Hint : Use the rules of addition of angular momenta to determine the total isospin of the final and initial states. There could be linear combinations !*

c) Isospin is an example of a flavor symmetry. Another one is the Eightfold Way, which is based on the group SU(3). What are the 5 lowest-dimensional irreps of this group ?

d) Since SU(3) has rank 2, we can construct two (conserved) quantum numbers, associated with the two generators from the Cartan subalgebra. We can again choose isospin as one of them, because SU(2) is a subgroup of SU(3). The other one is called strangeness. Using those two quantum numbers as axes in a two-dimensional space, one can assign many mesons and baryons to multiplets leading to those famous Eightfold Way patterns. Try to assign the nucleons and pions in such a manner, filling the "empty slots" with the corresponding baryons and mesons. We will do this together in class.

The objects that transform in the fundamental representation of SU(3), however, are not the hadrons themselves, but the three lightest quarks, u, d and s. Historically, this realization led to the development of the quark model. The up and down quarks have isospin $I_3 = 1/2$ and $I_3 = -1/2$, respectively, and strangeness 0, whereas the strange quark has strangeness $S = -1$ and isospin 0. It is often more useful to work with hypercharge Y (do not confuse this with the weak hypercharge!) than with strangeness. They are related by $Y = B+S$, where B is the baryon number, which is $B = 1/3$ for quarks and $B = -1/3$ for anti-quarks.

e) Recall from problem set 1 that the generators of SU(3) are given by (in the fundamental representation)

$$T_i = \frac{1}{2}\lambda_i,$$

where λ_i are the Gell-Mann matrices. The diagonal generators are T_3 and T_8. Show explicitly that the quark-states

$$|u> = (1, 0, 0) \tag{1}$$
$$|d> = (0, 1, 0) \tag{2}$$
$$|s> = (0, 0, 1) \tag{3}$$

are eigenstates of these operators and determine their eigenvalues. Convince yourself that T_3 corresponds to isospin, while $Y = \frac{2}{\sqrt{3}}T_8$ corresponds to hypercharge.

f) Historically, the so-called Gell-Mann - Nishijima formula

$$Q = T_3 + \frac{1}{2}(B + S)$$

was an experimental observation relating the charge Q of the hadrons to their other quantum numbers. Convince yourself that this formula is correct for the quarks and the hadrons from the previous questions.

g) In order to construct higher-dimensional irreps, we can proceed in a similar manner as in part a) for SU(2). Let us just consider the first step. We define the ladder operators according to

$$T_\pm = T_1 \pm iT_2 \tag{4}$$
$$V_\pm = T_4 \pm iT_5 \tag{5}$$
$$U_\pm = T_6 \pm iT_7. \tag{6}$$

Show how these operators act on a state $|T_3, Y>$ that carries the eigenvalues T_3 and Y (you can neglect the overall phase). *Hint : You should first calculate the commutators between the ladder operators and the operators* T_3 *and* Y.