a) $J_\pm |j, m_j\rangle = \sqrt{j(j+1) - m_j(m_j \pm 1)}\, |j, m_j \pm 1\rangle$

Spin 3/2 rep:

$j = 3/2 \qquad m_j = -3/2,\ -1/2,\ 1/2,\ 3/2.$

$|3/2, 3/2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$|3/2, 1/2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$|3/2, -1/2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$|3/2, -3/2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$J_3 |j, m_j\rangle = m_j |j, m_j\rangle$

$$\boxed{J_3 = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}}$$

$J_1 = ? \qquad J_2 = ?$

$J_\pm = J_1 \pm i J_2 \iff J_1 = \frac{1}{2}(J_+ + J_-)$

$J_2 = \frac{-i}{2}(J_+ - J_-)$

$J_+ = (\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$

$\hookrightarrow J_+ |3/2, 3/2\rangle = 0 \longrightarrow \vec{v}_1 = 0$

$\hookrightarrow J_+ |3/2, 1/2\rangle = \sqrt{\left(\frac{3}{2}\right)\left(\frac{5}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}\, |3/2, 3/2\rangle$

$= \sqrt{3}\, |3/2, 3/2\rangle \longrightarrow \vec{v}_2 = \begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\hookrightarrow J_+ |3/2, -1/2\rangle = \sqrt{\frac{15}{4} + \frac{1}{4}}\, |3/2, 1/2\rangle$

$= 2\, |3/2, 1/2\rangle \longrightarrow \vec{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

$$\hookrightarrow J_+ |3/2, -3/2\rangle = \sqrt{3}\, |3/2, -1/2\rangle \rightarrow \vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ \sqrt{3} \\ 0 \end{pmatrix}$$

$$J_+ = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$J_- = J_+^{\dagger} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{5} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$\hookrightarrow J_1 = \frac{1}{2}(J_+ + J_-)$$

$$\boxed{J_1 = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}}$$

$$J_2 = \frac{-i}{2}(J_+ - J_-)$$

$$\boxed{J_2 = \frac{1}{2}\begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -i2 & 0 \\ 0 & i2 & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}}$$

$s = j$ - representation $\longrightarrow$ $(2j+1)$-(dim) representation.

$j = \frac{1}{2}$ $\longrightarrow \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix}$ 2 dim* rep of $SU(2)$*

$j = 1$ $\longrightarrow \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ 3 rep of $SU(2)$**

* $\hookrightarrow$ fundamental rep of $SU(2)$

** $\hookrightarrow$ Adjoint rep of $SU(2)$

$$\mathcal{U} = e^{-i\theta_a \tau_a}, \quad a = 1, 2, 3$$

$$A \longrightarrow \mathcal{U} A \mathcal{U}^{\dagger}$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad e^{i v_\mu \sigma^\mu}$$

$$A = \frac{1}{2}\begin{pmatrix} v_0 + v_3 & v_1 - i v_2 \\ v_1 + i v_2 & v_0 - v_3 \end{pmatrix} = \frac{\mathrm{Tr}A}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to 1$$

$$+\frac{1}{2}\begin{pmatrix} v_3 & v_1 - i v_2 \\ v_1 + i v_2 & -v_3 \end{pmatrix} \to 3$$

$$\leftrightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\mathrm{Tr}A = v_0$$

(b) $\overset{3}{(\pi^+, \pi^0, \pi^-)}$ $\quad \overset{2}{(p, n)}$

$$|p\rangle = |1/2, 1/2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$|n\rangle = |1/2, -1/2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\pi^+\rangle = |1, 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$|\pi^0\rangle = |1, 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$|\pi^-\rangle = |1, -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(A) $\pi^+ + p \longrightarrow \pi^+ + p$

$$|\pi^+\rangle \otimes |p\rangle = |j = 3/2, m_j = 3/2\rangle \equiv |3/2, 3/2\rangle_j$$
$$|1, 1\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$${}_j\langle 3/2, 3/2 | S | 3/2, 3/2\rangle_j = \mathcal{M}_A$$

$${}_j\langle 3/2, \bullet | S | 3/2, \bullet\rangle_j = \mathcal{M}_3 \to \mathcal{M}_3 = \mathcal{M}_A$$

$${}_j\langle 1/2, \bullet | S | 1/2, \bullet\rangle_j = \mathcal{M}_1$$

independent of m_j (because of $SU(2)$-sym.)

(B) $\pi^0 + p \longrightarrow \pi^0 + p$.

$$|\pi^0, p\rangle \equiv |\pi^0\rangle \otimes |p\rangle = \sqrt{\tfrac{2}{3}}\,|3/2, 1/2\rangle_j - \sqrt{\tfrac{1}{3}}\,|1/2, 1/2\rangle_j$$

$|1,0\rangle \otimes |1/2, 1/2\rangle$

Clebsch - Gordon coefficients.

$$\mathcal{M}_B = \langle \pi^0, p | S | \pi^0, p \rangle$$

$$= \frac{2}{3}\mathcal{M}_3 + \frac{1}{3}\mathcal{M}_1 = \frac{1}{3}\left(2\underset{\mathcal{M}_A}{\overset{\parallel}{\mathcal{M}_3}} + \mathcal{M}_1\right)$$

(C) $\pi^0 + p \longrightarrow \pi^+ + n$

$$|\pi^+, n\rangle = \frac{1}{\sqrt{3}}\,|3/2, 1/2\rangle_j + \sqrt{\tfrac{2}{3}}\,|1/2, 1/2\rangle_j$$

$|1,1\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$

$$\mathcal{M}_C = {}_j\langle \pi^+, n | S | \pi^0, p \rangle_j$$

$$= \frac{\sqrt{2}}{3}\mathcal{M}_3 - \frac{\sqrt{2}}{3}\mathcal{M}_1 = \frac{\sqrt{2}}{3}(\mathcal{M}_3 - \mathcal{M}_1)$$

(D) $\pi^0 + n \longrightarrow \pi^- + p$.

$$|\pi^0, n\rangle = \sqrt{\tfrac{2}{3}}\,|3/2, -1/2\rangle_j + \frac{1}{\sqrt{3}}\,|1/2, -1/2\rangle_j$$

$$|\pi^-, p\rangle = \frac{1}{\sqrt{3}}\,|3/2, -1/2\rangle_j - \sqrt{\tfrac{2}{3}}\,|1/2, -1/2\rangle_j$$

$$\mathcal{M}_D \equiv \langle \pi^-, p | S | \pi^0, n \rangle$$

$$= \frac{\sqrt{2}}{3}\mathcal{M}_3 - \frac{\sqrt{2}}{3}\mathcal{M}_1 = \frac{\sqrt{2}}{3}(\mathcal{M}_3 - \mathcal{M}_1)$$

$$\hookrightarrow \boxed{\mathcal{M}_C = \mathcal{M}_D}$$

$$\mathcal{M}_A = \mathcal{M}_3$$
$$\mathcal{M}_B = 2/3\, \mathcal{M}_3 + 1/2\, \mathcal{M}_1$$
$$\mathcal{M}_C = \sqrt{2}/3\, \mathcal{M}_3 - \sqrt{2}/3\, \mathcal{M}_1$$
$$\mathcal{M}_D = \sqrt{2}/3\, \mathcal{M}_3 - \sqrt{2}/3\, \mathcal{M}_1$$

d) $SU(3)$ flavor symmetry.

1, $\underset{\text{fundamental}}{3}$ (q), $\underset{\text{(anti-fundamental)}}{\bar{3}}$ $(\bar{q})$

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \longrightarrow q' = \mathcal{U} q \quad , \mathcal{U} \in SU(3)$$
$$q'_a = \mathcal{U}_{ab} q_b$$

$$\bar{q} = (\bar{q}_1, \bar{q}_2, q_3) \longrightarrow \bar{q}' = q\, \mathcal{U}^\dagger$$
$$\bar{q}'_{\bar{a}} = \bar{q}_b\, \mathcal{U}^\dagger_{b\bar{a}}$$

$(2 \times \bar{2}) = 3 + \underset{\hookrightarrow \mathrm{Tr}\,\pi}{1}$ (in $SU(2)$)

$$3 \times \bar{3} = \underset{\substack{\uparrow \\ \text{Adjoint} \\ \hookrightarrow = \lambda_a T^a}}{8} + \underset{\substack{\uparrow \\ \text{singlet}}}{1} \hookrightarrow \mathrm{Tr}\, m$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \otimes (\bar{q}_1\, \bar{q}_2\, \bar{q}_3)$$

$$q_a \times \bar{q}_{\bar{a}} = (q\bar{q})_{a\bar{a}} \equiv m_{a\bar{a}} \qquad \mathrm{tr}\, m = (q\bar{q})_{a\bar{a}}\, \delta_a^{\bar{a}}$$
$$\hookrightarrow m = \underbrace{\frac{\mathrm{Tr}(m)}{3}}_{1} + \underbrace{\lambda_a \tau_a}_{8}$$

$$\underset{q}{3_a} \times \underset{q}{3_a} = \underset{\substack{\downarrow \\ \text{Symmetric}}}{6} + \underset{\substack{\downarrow \\ \text{Anti-Symmetric}}}{\bar{3}}$$

$$qq \equiv T = \underbrace{\frac{1}{2}(T + T^T)}_{S} + \underbrace{\frac{1}{2}(T - T^T)}_{A}$$

$$\begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & s_4 & s_5 \\ s_3 & s_5 & s_6 \end{pmatrix} + \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix} \leftarrow T_{ij} = T_{(ij)} + T_{[ij]}$$

$$\underset{q}{3} \times (\underset{q}{3} \times \underset{q}{3}) = 3 \times (6 + \bar{3}) = (3 \times 6) + (3 \times \bar{3}) = 10 + 8 + 8 + 1$$

Q: How does $3 \times \bar{3}$ transform?

A:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \otimes \begin{pmatrix} \bar{q}_{\bar{1}} & \bar{q}_{\bar{2}} & \bar{q}_{\bar{3}} \end{pmatrix} = \begin{pmatrix} q_1 \bar{q}_{\bar{1}} & q_1 \bar{q}_{\bar{2}} & q_1 \bar{q}_{\bar{3}} \\ q_2 \bar{q}_{\bar{1}} & q_2 \bar{q}_{\bar{2}} & q_3 \bar{q}_{\bar{2}} \\ q_3 \bar{q}_{\bar{1}} & q_3 \bar{q}_{\bar{2}} & q_3 \bar{q}_{\bar{3}} \end{pmatrix} \equiv M$$

$$q \longrightarrow U q$$

$$\bar{q} \longrightarrow \bar{q} U^{+}$$

$$q\bar{q} \longrightarrow U \underbrace{q \bar{q}}_{M} U^{+} \equiv U \underbrace{M}_{} U^{+}$$

$$M = \underbrace{m_a \tau^a}_{8} + \underbrace{\left(\frac{\mathrm{Tr} M}{3}\right)}_{1}$$

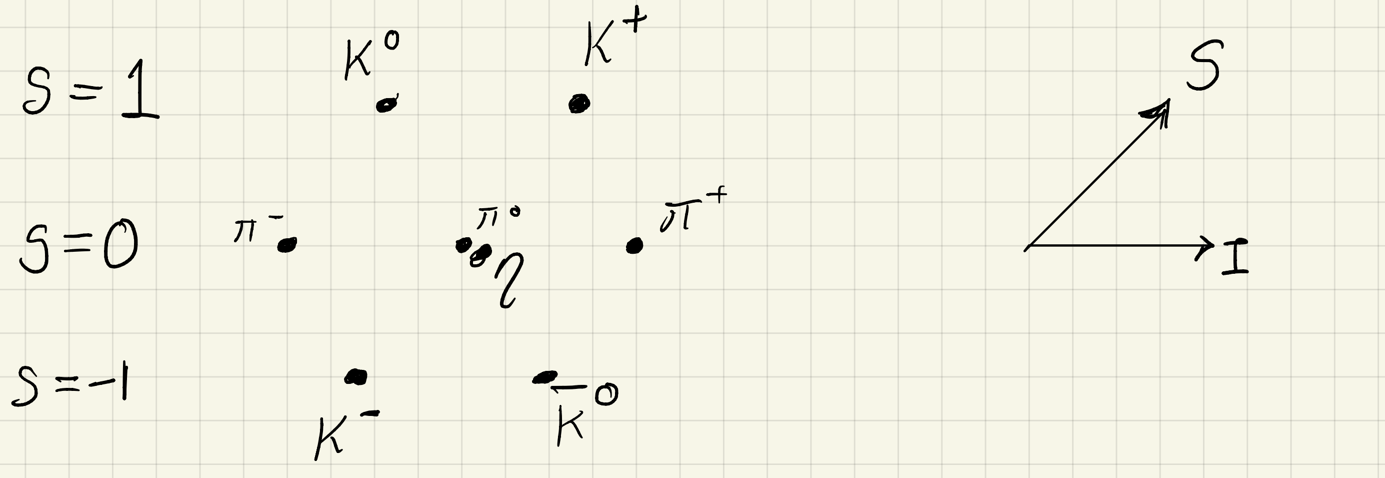
$$8 \downarrow$$

$$\begin{pmatrix} m_3 + \frac{m_8}{\sqrt{3}} & m_1 - i m_2 & m_4 - i m_5 \\ m_1 + i m_2 & -m_3 + \frac{m_8}{\sqrt{3}} & m_6 - i m_7 \\ m_4 + i m_5 & m_6 - i m_7 & -2 m_8 \end{pmatrix}$$

c) $\rightarrow$ 1, $3\,(\bar{3})$, $6\,(\bar{6})$, 8, $10\,(\overline{10})$

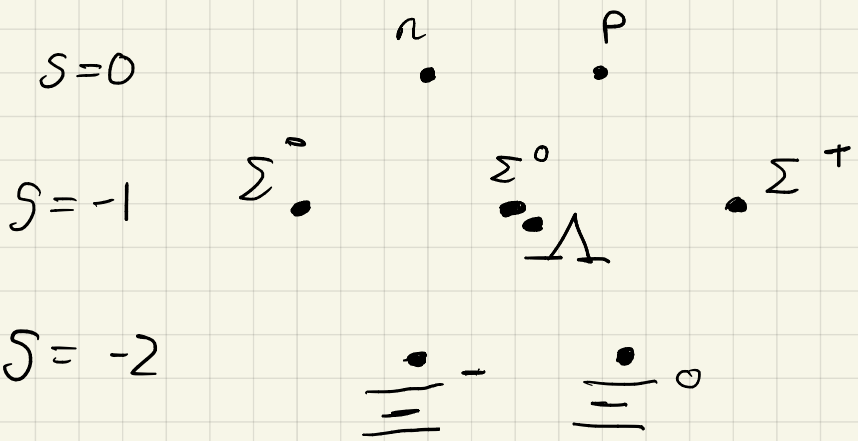
Singlet, (anti) fundamental, symmetric, Adjoint, symmetric

d) The pions belong the Meson octet (8)



↳ We introduced four Kaons that carry strangeness and the η-meson that is an isospin singlet.

(*) The nucleons belong to the baryon octet.



Eight-fold Way

e) $T_i = \frac{1}{2}\lambda_i$, Gell-Mann matrices.

$$T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{Tr}\, T_i T_j = \frac{1}{2}\delta_{ij}$$

$$T_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$|d\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$|s\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$Y = \frac{2}{\sqrt{3}} T_8$$

$$\underset{\substack{\text{Hypercharge} \\ \text{number}}}{Y} = \underset{\substack{\text{Baryon} \\ \text{number}}}{\overset{\frac{1}{3}\text{ quarks}}{B}} + \underset{\text{Strangeness}}{S}$$

$\cdot)\quad T_3|u\rangle = \frac{1}{2}|u\rangle \qquad T_8|u\rangle = \frac{1}{2\sqrt{3}}|u\rangle$

$$\hookrightarrow Y|u\rangle = \frac{1}{3}|u\rangle$$

$\cdot)\quad T_3|d\rangle = -\frac{1}{2}|d\rangle \qquad T_8|d\rangle = \frac{1}{2\sqrt{3}}|d\rangle$

$$\hookrightarrow Y|d\rangle = \frac{1}{3}|d\rangle$$

$\cdot)\quad T_3|s\rangle = 0\cdot|s\rangle$

$T_8|s\rangle = -\frac{1}{\sqrt{3}}|s\rangle \qquad Y|s\rangle = -\frac{2}{3}|s\rangle$

$$\hookrightarrow Y = \frac{2}{\sqrt{3}}T_8 .$$

---

f) $$Q = T_3 + \frac{1}{2}(B+S)$$

Quarks:

$Q(u) = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{3}+0\right) = \frac{2}{3}$ ✓

$Q(d) = -\frac{1}{2} + \frac{1}{2}\left(\frac{1}{3}+0\right) = -\frac{1}{3}$ ✓

$Q(s) = 0 + \frac{1}{2}\left(\frac{1}{3}-1\right) = \frac{-1}{3}$ ✓

Mesons: $(q\bar{q}) \to B=0$

$Q(\pi^-) = -1 + \frac{1}{2}(0+0) = -1$

$Q(\pi^0) = 0 + \frac{1}{2}(0+0) = 0 \to \pi^0$ is E.M. neutral

$Q(\pi^+) = 1 + \frac{1}{2}(0+0) = 1$

$$Q(\eta) = 0 + 0 = 0$$

$$Q(K^0) = -\frac{1}{2} + \frac{1}{2}(0+1) = 0$$

$$Q(K^+) = \frac{1}{2} + \frac{1}{2}(0+1) = 1$$

$$Q(K^-) = -1$$

$$Q(\bar{K}^0) = 0$$

e.g. $Q(K^0) = Q(d\bar{s}) = \left(-\frac{1}{2}+0\right) + \frac{1}{2}\left(\frac{1}{3}+\frac{2}{3}\right) = 0$

$Q(K^-) = Q(\bar{u}s) = \left(-\frac{1}{2}+0\right) + \frac{1}{2}\left(-\frac{1}{3}-\frac{2}{3}\right) = -1$

Baryons: $B=1$ (qqq)

$$Q(\Sigma^-) = -1 + \frac{1}{2}(1-1) = -1 \quad \checkmark$$

$$Q(\Sigma^0) = T_3(\Sigma^0) = 0 \quad \checkmark$$

$$Q(\Sigma^+) = 1 \quad \checkmark$$

$$Q(n) = -\frac{1}{2} + \frac{1}{2}(1+0) = 0$$

$$Q(p) = \frac{1}{2} + \frac{1}{2}(1+0) = 1$$

↳ ...

g)
$$T_\pm = T_1 \pm i T_2$$
$$V_\pm = T_4 \pm i T_5$$
$$U_\pm = T_6 \pm i T_7$$

$|T_3, Y\rangle$

$T_\pm |T_3, Y\rangle = ?$

Hint: $[T_3, T_\pm] = [T_3, T_1 \pm i T_2]$

$$= [T_3, T_1] \pm i [T_3, T_2]$$

$$= -i f_{13a} T_a \mp i \cdot i f_{23a} T_a$$

$$= -i \underbrace{f_{132}}_{=-1} T_2 \pm \underbrace{f_{231}}_{=1} T_1$$

$$= i T_2 \pm T_1$$

$$= \pm T_\pm$$

$$[T_3, V_\pm] = \underbrace{[T_3, T_4]}_{i \underbrace{f_{34a}}\, T_a = \frac{i}{2} T_5} \pm i \underbrace{[T_3, T_5]}_{\underbrace{i f_{35a} T_a}_{= -\frac{i}{2} T_4}}$$

$$= \frac{1}{2} (i T_5 \pm T_4) = \pm \frac{1}{2} V_\pm$$

$$[T_3, U_\pm] = \mp \frac{1}{2} U_\pm$$

$$[Y, T_\pm] = \frac{2}{\sqrt{3}} [T_8, T_1 \pm i T_2]$$

$$= 0$$

$$[Y, V_\pm] = \pm V_\pm$$

$$[Y, U_\pm] = \pm U_\pm$$

$$\hookrightarrow \hat{T}_3 \hat{T}_\pm |T_3, y\rangle = \hat{T}_\pm \hat{T}_3 |T_3, y\rangle + \underbrace{[\hat{T}_3, \hat{T}_\pm]}_{= \pm \hat{T}_\pm} |T_3, y\rangle$$

$$= \hat{T}_\pm (T_3 \pm 1) |T_3, y\rangle$$

$$= (T_3 \pm 1) \hat{T}_\pm |T_3, y\rangle$$

$$\boxed{\hat{T}_\pm |T_3, y\rangle \sim |T_3 \pm 1, y\rangle}$$

- $\hat{y} \hat{T}_\pm |T_3, y\rangle = y \hat{T}_\pm |T_3, y\rangle$

- $\hat{T}_3 \hat{V}_\pm |T_3, y\rangle = (T_3 \pm 1/2) \hat{V}_\pm |T_3, y\rangle$

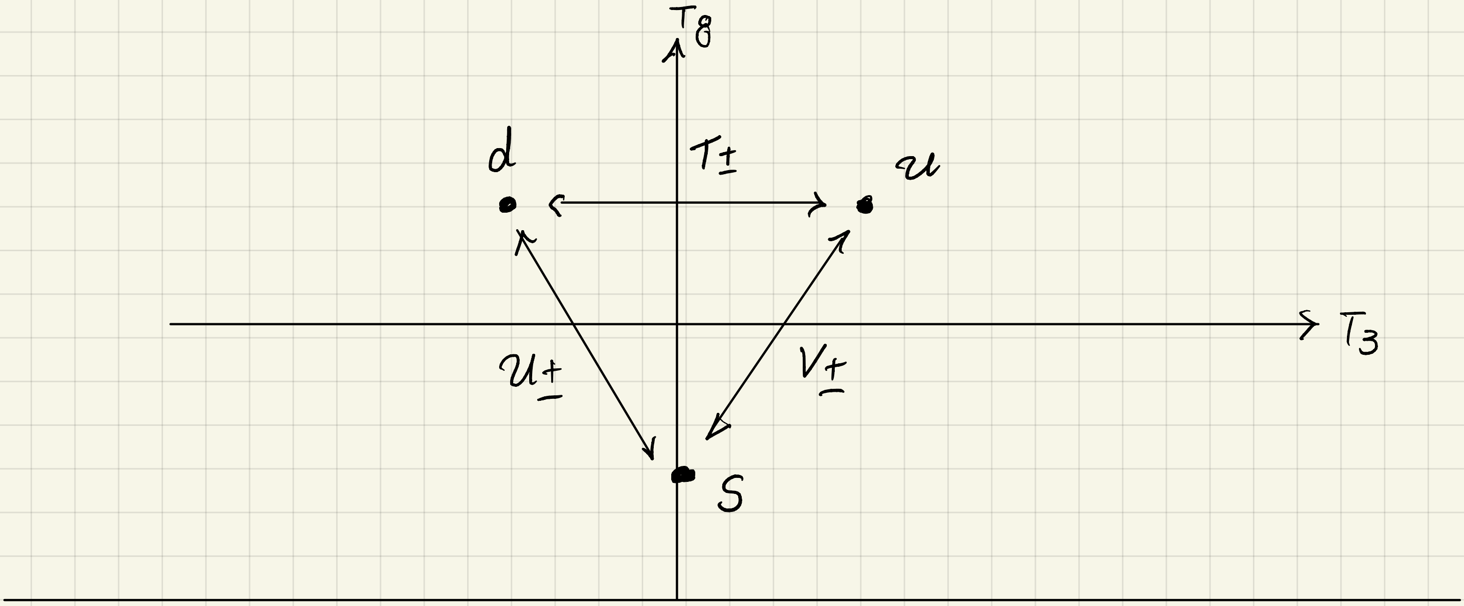
$\hat{y} \hat{V}_\pm |T_3, y\rangle = (y \pm 1) \hat{V}_\pm |T_3, y\rangle$

$$\hookrightarrow \boxed{\hat{V}_\pm |T_3, y\rangle \sim |T_3 \pm \tfrac{1}{2}, y \pm 1\rangle}$$

- $\hat{T}_3 \hat{U}_\pm |T_3, y\rangle = (T_3 \mp \frac{1}{2}) \hat{U}_\pm |T_3, y\rangle$

$\hat{y} \hat{U}_\pm |T_3, y\rangle = (y \pm 1) \hat{U}_\pm |T_3, y\rangle$

$$\hookrightarrow \boxed{\hat{U}_\pm |T_3, y\rangle \sim |T_3 \mp \tfrac{1}{2}, y \pm 1\rangle}$$



Q: • π^0, η are not degenerate states

↳ Why?

A: $\boxed{\begin{matrix} I(\pi^0) = 1 \\ I(\eta) = 0 \end{matrix}}$ $\quad$ $I_3(\pi^0) = 0$

$I_3(\eta) = 0$