Chiral symmetry breaking and sigma model.

a) QCD & massless quarks (up & down)

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + \bar{u} i \not{D} u + \bar{d} i \not{D} d.$$

where $D_\mu = \partial_\mu - i g A^a_\mu T^a$

$$Q \equiv \begin{pmatrix} u \\ d \end{pmatrix} \rightsquigarrow Q = Q_L + Q_R$$
$$= \begin{pmatrix} u \\ d \end{pmatrix}_L + \begin{pmatrix} u \\ d \end{pmatrix}_R.$$

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + \bar{Q}_L i \not{D} Q_L + \bar{Q}_R i \not{D} Q_R$$

$Q_L \longrightarrow \mathcal{U}_L Q_L$ where $\underbrace{\mathcal{U}_L, \mathcal{U}_R}_{\text{independent!}} \in \mathcal{U}(2)$

$Q_R \longrightarrow \mathcal{U}_R Q_R$

↳ Global symmetry.

$$\mathcal{U}(2)_L \times \mathcal{U}(2)_R \simeq \underbrace{SU(2)_L \times SU(2)_R}_{\text{Non-Abelian}} \times \underbrace{\mathcal{U}(1)_L \times \mathcal{U}(1)_R}_{\text{Abelian}}$$

The abelian part can be rewritten as:

$\mathcal{U}(1)_L \to Q_L \longrightarrow e^{i\alpha} Q_L$ $\longleftrightarrow$ $Q_L \to e^{i\alpha} Q_L$ & $Q_L \to e^{i\beta} Q_L$

$\mathcal{U}(1)_R \to Q_R \longrightarrow e^{i\beta} Q_R$ $\quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \to \begin{pmatrix} \alpha + \beta \\ \alpha \ \beta \end{pmatrix}$ $\quad \underbrace{Q_R \to e^{i\alpha} Q_R}_{} \quad \underbrace{Q_R \to e^{-i\beta} Q_R}_{}$

$$\underbrace{Q \to e^{i\alpha} Q}_{\mathcal{U}(1)_V} \qquad \underbrace{Q \to e^{i\beta\gamma_5} Q}_{\mathcal{U}(1)_A}$$

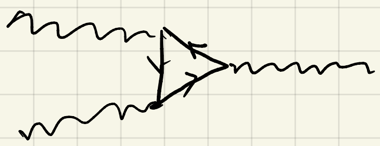
$$\mathcal{U}(1)_L \times \mathcal{U}_R = \underbrace{\mathcal{U}(1)_V}_{\substack{\text{Baryon}\\ \text{Number}\\ (B)}} \times \underbrace{\mathcal{U}(1)_A}_{?}$$

↳ origin of the "$\mathcal{U}(1)_A$ problem" (Weinberg '75).

$\mathcal{U}(1)_A$ is anomalous:

$$Z = \int \underbrace{[D\psi]}_{\substack{\text{is not}\\ \mathcal{U}(1)_A \text{ invariant.}}} e^{i\overbrace{S[\psi]}}$$

$$S[\psi] = \int dx^\mu \, \mathcal{L}[\psi]$$

↳ 

↳ $\mathcal{U}(1)_V$ is a <u>good</u> symmetry of QCD

- N massless quarks:

$$\mathcal{U}(N)_L \times \mathcal{U}(N)_R = SU(N)_L \times SU(N)_R \times \mathcal{U}(1)_V \times \mathcal{U}(1)_A$$

b) Low energy QCD is not chiral.

↳ Symmetry must broken → Spontaneously.

$$\underbrace{SU(2)_L}_{3 \text{ generators}} \times \underbrace{SU(2)_R}_{3 \text{ generators}} \xrightarrow{?} \underbrace{SU(2)_V}_{\substack{3 \text{ generator}\\ +3 \text{ Goldstone boson}}} \rightarrow (\text{Pions})$$

$SU(2)_V$: $\quad Q_L \to \mathcal{U}\, Q_L \quad : \mathcal{U}_L = \mathcal{U}_R = \mathcal{U} \in SU(2)_V$

$\quad\quad\quad\quad Q_R \to \mathcal{U}\, Q_R$

$$\underbrace{SU(3)_L}_{8 \text{ generators}} \times \underbrace{SU(3)_R}_{8 \text{ generators}} \xrightarrow{?} \underbrace{SU(3)_V}_{\substack{8 \text{ generator} \\ + \\ 8\text{ G.B}}}$$

$8 \text{ G.B} \to \underbrace{\begin{pmatrix} \text{Kaons} \\ \text{Pions} \\ \eta \end{pmatrix}}_{\text{"Eight fold way"}}$

Order parameter $\Rightarrow$? ϕ:

Criteria: - Lorentz-scalar

- $SU(3)_c$-singlet
- $SU(2)_V$-invariant. ✓ To be checked.

↳ We must use a composite field.

$$\underbrace{\langle \bar{u} u \rangle}_{\text{Chiral condensate}} = \langle \bar{d} d \rangle = v^3 \quad \text{or} \quad \langle \bar{Q}^j Q^i \rangle = v^3 \delta^{ij}$$

↳ Chiral condensate.

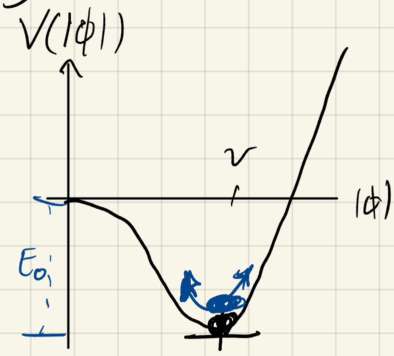
c) Goal: Find low-energy EFT $\mathcal{L}[\phi]$

$$\mathcal{L} = \frac{1}{2} |\partial_\mu \phi|^2 + \frac{\mu^2}{2} |\phi|^2 - \frac{\lambda}{4} (|\phi|^2)^2 + E_0$$

where $|\phi|^2 \equiv \mathrm{tr}(\phi^\dagger \phi)$

$$\frac{dV}{d|\phi|} = (-\mu^2 + \lambda |\phi|^2)|\phi| \overset{!}{=} 0$$

$$|\phi| = \sqrt{\frac{\mu^2}{\lambda}} \equiv v$$



Rewrite $\mathcal{L} = \frac{1}{2}|\partial_\mu \phi|^2 - \frac{\lambda}{4}\left(|\phi|^2 - v^2\right)^2$

$$2 \times \bar{2} = 1 + 3 = \frac{1}{\sqrt{2}}\left(\overbrace{\mathrm{tr}\, M}^{\sigma}\, \mathbb{1} + i\pi^a \sigma^a\right)$$

$$\phi^{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma + i\pi^3 & -\pi^2 + i\pi^1 \\ \pi^2 + i\pi^1 & \sigma - i\pi^3 \end{pmatrix}$$

$$\mathrm{tr}(\phi^\dagger \phi) = \frac{1}{2}\,\mathrm{tr}\begin{pmatrix} \sigma - i\pi^3 & -\pi^2 - i\pi^1 \\ \pi^2 - i\pi^1 & \sigma + i\pi^3 \end{pmatrix}\begin{pmatrix} \sigma + i\pi^3 & -\pi^2 + i\pi^1 \\ \pi^2 + i\pi^1 & \sigma - i\pi^3 \end{pmatrix}$$

$$= \frac{1}{2}\,\mathrm{tr}\begin{pmatrix} \sigma^2 + \pi^a\pi^a & 0 \\ 0 & \sigma^2 + \pi^a\pi^a \end{pmatrix}$$

$$= \sigma^2 + \pi^a \pi^a$$

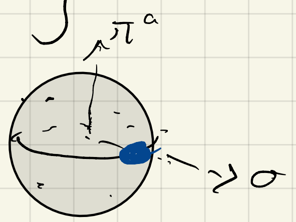
↳ Vacuum Manifold:

$$\mathcal{M} = \{(\sigma, \pi^a) \,/\, \underbrace{V(|\phi|) = 0}_{\frac{\lambda}{4}\left((\sigma^2 + \pi^a\pi^a) - v^2\right)^2 = 0}\}$$

$$\Updownarrow$$

$$\mathcal{M} = \{(\sigma, \pi^a) \,/\, \sigma^2 + \pi^a \pi^a = v^2\}$$

$$\mathcal{M} \rightsquigarrow S^3$$



Lets choose $\sigma = v$, $\pi^a = 0$ as the ground state and expand $\mathcal{L}$ around it:

$$\phi^{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix} (v + \sigma) + i\pi^3 & -\pi^2 + i\pi^1 \\ \pi + i\pi^3 & (v + \sigma) - i\pi^3 \end{pmatrix}$$

↳ $\frac{1}{2}|\partial_\mu \phi|^2 = \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a$

↳ $\frac{\lambda}{4}(|\phi|^2 - v^2)^2 = \frac{\lambda}{4}((v+\sigma)^2 + \pi^a\pi^a - v^2)^2$

$$= \frac{\lambda}{4}(\sigma^2 + 2v\sigma + \pi^a\pi^a)^2$$

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a - \frac{\lambda}{4}(\sigma^2 + 2v\sigma + \pi^a\pi^a)^2}$$

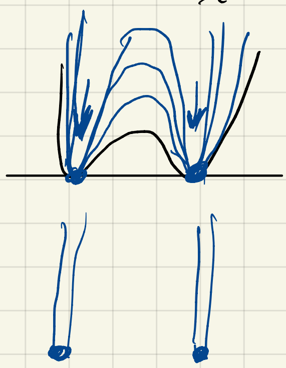
↳ The Linear sigma-model

↳ "The axial vector Current in β-decay" ('60)

↳ It is renormalizable.

↳ The non-Linear sigma model.

Double scaling Limit: $\lambda \to \infty$, $\mu^2 \to \infty$ s.t. $v^2 = \frac{\mu^2}{\lambda}$ is fixed.



$$V(|\phi|) = \frac{\lambda}{4}(|\phi|^2 - v^2)^2 \longrightarrow \begin{cases} \infty, & |\phi|^2 \neq v^2 \\ 0, & |\phi|^2 = v^2 \end{cases}$$

↓

it becomes a constraint!

↳ $\underbrace{\pi^a\pi^a + \sigma^2 = -2\sigma v}$. (Understood as com for λ)

$$\sigma_\pm = -v \pm \sqrt{v^2 - \pi^a\pi^a}$$

$$\sigma = \sigma(\pi^a)$$

$$\partial_\mu \sigma_\pm = \pm \frac{-\partial_\mu(\pi^a\pi^a)}{\sqrt{v^2-\pi^a\pi^a}} = \mp \frac{\pi^a\partial_\mu\pi^a}{\sqrt{v^2-\pi^a\pi^a}}$$

$$\hookrightarrow \boxed{\mathcal{L} = \frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^a + \frac{1}{2}\frac{\pi^a\partial_\mu\pi^a\,\pi^b\partial^\mu\pi^b}{v^2 - \pi\pi^a}} \to$$

The non-Linear sigma model

Lets expan $\mathcal{L}$: (around $\pi^a = 0$, i.e. $\pi^a\pi^a \ll v^2$)

Low-energy QCD

$$(v^2 - \pi^a\pi^a)^{-2} = v^{-2}\left(1 + \frac{\pi^a\pi^a}{v^2} + O\left((\pi^a\pi^a)^2/v^4\right)\right)$$

$$\hookrightarrow \mathcal{L} = \frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^a + \boxed{\frac{1}{2v^2}(\pi^a\partial_\mu\pi^a)(\pi^b\partial^\mu\pi^b)\Big(1} + \underbrace{\frac{\pi^a\pi^a}{v^2}\Big) + O(\pi^8)}_{O(\pi^6)}$$

The EOM for π^a:

$$\hookrightarrow \Box\pi^a \cancel{- \frac{1}{v^2}\pi^b\partial_\mu\pi^b}\,\partial^\mu\pi^a + \frac{1}{v^2}\underbrace{\partial_\mu\left(\pi^b\partial^\mu\pi^b\,\pi^a\right)}_{} = 0$$

$$= \frac{1}{v^2}\partial_\mu\pi^b\partial^\mu\pi^b\,\pi^a + \frac{1}{v^2}\pi^b\Box\pi^b\,\pi^a$$

$$\cancel{+ \frac{1}{v^2}\pi^b\partial^\mu\pi^b\,\partial_\mu\pi^a}$$

$$\hookrightarrow \Box\pi^a = \frac{(\partial_\mu\pi^b)^2\pi^a}{v^2} + \frac{\pi^b\Box\pi^b\,\pi^a}{v^2}$$

Now, Lets start from the given expression:

$$\mathcal{L} \supset \frac{1}{6v^2}\left((\pi^a \partial_\mu \pi^a)^2 - \pi^a \pi^a \partial_\mu \pi^b \partial^\mu \pi^b\right)$$

$$= \frac{1}{6v^2}\left((\pi^a \partial_\mu \pi^a)^2 + \underbrace{\pi^b \partial_\mu(\pi^a \pi^a \partial^\mu \pi^b)}_{= 2(\pi^a \partial_\mu \pi^a)^2 + \pi^a \pi^a \pi^b \Box \pi^b} \underbrace{- \partial_\mu(\pi^a \pi^a \pi^b \partial^\mu \pi^b)}_{\text{Boundary term}}\right)$$

$$= 2(\pi^a \partial_\mu \pi^a)^2 \underbrace{+ \pi^a \pi^a \pi^b \left(\frac{(\partial_\mu \pi^a)^2 \pi^b}{v^2} + \frac{\pi^c \Box \pi^c \pi^b}{v^2}\right)}_{\in \mathcal{O}(\pi^6)}$$

$$\mathcal{L} = \frac{1}{6v^2}\left((\pi^a \partial_\mu \pi^a)^2 + 2(\pi^a \partial_\mu \pi^a)^2\right)$$

$$\boxed{= \frac{1}{2v^2}(\pi^a \partial_\mu \pi^a)^2}$$

f) Exponential Representation

$$\underset{\text{Complex field}}{\phi} = \frac{\rho}{\sqrt{2}} e^{i\theta/v} \qquad \theta \to \theta + a$$

$\rho \to$ real $\to$ Massive

$\theta \to$ real $\to$ Massless $\to$ G.B.

$$\underset{\in \text{ Lie group } SU(2)}{\mathcal{U}} = \exp\left(i \frac{\pi^a \overset{\in \text{ Lie algebra } su(2)}{\sigma^a}}{\underset{\text{Pion decay constant.}}{f_\pi}}\right)$$

$$\mathcal{L}_{Chiral} = \frac{v^2}{4} \mathrm{Tr}\left(\partial_\mu \mathcal{U}^\dagger \partial^\mu \mathcal{U}\right)$$

$$\mathcal{U} \rightarrow L\,\mathcal{U}R^\dagger, \quad \underbrace{L, R}_{\text{independent!}} \in SU(2)$$

$$\delta^a \pi^c \longrightarrow i f^{abc} \frac{\pi^b}{v}.$$

g) • $\mathcal{U}^\dagger \mathcal{U} = \mathbb{1}$

• $\theta\mathcal{U}$. can be eliminated by redefining $\mathcal{U}$.

↳ Thus we can organise $\mathcal{L}_{chiral}$ by $\# \partial$.

h) $\delta \mathcal{L}_{mass} = v^3\, \mathrm{tr}\left(M\mathcal{U} + M^\dagger \mathcal{U}^\dagger\right)$

$$\left.\begin{aligned} M &\rightarrow R\,M\,L^\dagger \\ \mathcal{U} &\rightarrow L\,\mathcal{U}R^\dagger \end{aligned}\right\} \mathrm{tr}(M\mathcal{U}) \rightarrow \mathrm{tr}(M\mathcal{U})$$

$$\mathcal{L}_D = \bar{\psi}_L i\partial\!\!\!/ \psi_L + \bar{\psi}_R i\partial\!\!\!/ \psi_R - M \bar{\psi}_L \psi_R - M^* \bar{\psi}_R \psi_L$$

$$M = m e^{i\theta}$$

$$\underbrace{\begin{aligned} \psi_L &\rightarrow e^{i\theta/2} \psi_L \\ \psi_R &\rightarrow e^{-i\theta/2} \psi_R. \end{aligned}}_{\in \mathcal{U}(1)_A}$$

↳ It induces a term $\delta\mathcal{L}_\theta = \theta \frac{g^2}{32\pi^2} G^a_{\mu\nu}\tilde{G}^{\mu\nu a}$

Lets set $\theta = 0$ $\left(|\theta| < 10^{-9}\right)$

↳ $\mathcal{U} = \exp\left(\frac{i\pi^a\sigma^a}{f_\pi}\right) = 1 + \frac{i\pi^a\sigma^a}{f_\pi} - \frac{\pi^a\pi^b\sigma^a\sigma^b}{2f_\pi^2} + \dots$

↳ $v^3 \operatorname{tr}\left[M(\mathcal{U}+\mathcal{U}^+)\right]$

$$= v^3 \operatorname{tr}\Big[M\Big(1 + \frac{i\pi^a\sigma^a}{f_\pi} - \frac{\pi^a\pi^b\sigma^a\sigma^b}{2f_\pi^2} + \dots\Big)$$

$$+ 1 - \frac{i\pi^a\sigma^a}{f_\pi} - \frac{\pi^a\pi^b}{2f_\pi^2}\underbrace{\sigma^a\sigma^b}_{\delta^{ab}\mathbb{1} + \varepsilon^{abc}\sigma^c} + \dots\Big)$$

$$= -\frac{v^3}{f_\pi^2}\operatorname{tr}(M)\,\pi^a\pi^a$$

↳ $\boxed{m^2 = \frac{2v^3}{f_\pi^2}(m_u + m_d)}$ → Gell-Mann-Oakes-Renner relation.