① Flavor parameters & CKM matrix

The Yukawa sector of the Standard Model reads

$$\mathcal{L}_Y = -\Lambda^{(e)}_{ij} \bar{E}^i_L H e^j_R - \Lambda^{(d)}_{ij} \bar{Q}^i_L H d^j_R - \Lambda^{(u)}_{ij} \bar{Q}^i_L \tilde{H} u^j_R + H.c.$$

where $\tilde{H} = i\sigma_2 H^*$, the Λ's are Yukawa matrices, and i,j are family indices.

(a) Let's focus on the Yukawa interaction terms for the quarks

$$\mathcal{L}^{(q)}_Y = -\Lambda^{(d)}_{ij} \bar{Q}^i_L H d^j_R - \Lambda^{(u)}_{ij} \bar{Q}^i_L \tilde{H} u^j_R + H.c.$$

- Note that the Yukawa matrices $\Lambda^{(d)}_{ij}$ and $\Lambda^{(u)}_{ij}$ are general complex matrices.

- After symmetry breaking the quarks acquire masses

$$-\Lambda^{(d)}_{ij} \bar{Q}^i_L \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} d^j_R - \Lambda^{(u)}_{ij} \bar{Q}^i_L \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} u^j_R + H.c. =$$

$$= -\frac{v}{\sqrt{2}} \Big[\underbrace{\Lambda^{(d)}_{ij} \bar{d}^i_L d^j_R}_{\equiv \bar{d}_L \Lambda_d d_R} + \underbrace{\Lambda^{(u)}_{ij} \bar{u}^i_L u^j_R}_{\equiv \bar{u}_L \Lambda_u u_R} + H.c. \Big]$$

↳ Since $\Lambda_{d,u} \Lambda^\dagger_{d,u}$ is Hermitian, we can always find a diagonal matrix $M_{d,u}$ such that $\Lambda_{d,u} \Lambda^\dagger_{d,u} = U_{d,u} M^2_{d,u} U^\dagger_{d,u}$, with the unitary matrices $U_{d,u}$.

↳ $\Lambda_{d,u}$ itself does not have to be Hermitian, however we can generically write $\Lambda_{d,u} = U_{d,u} M_{d,u} K^\dagger_{d,u}$, with another unitary matrix $K_{d,u}$.

$$\Rightarrow \mathcal{L}^{(q)}_Y \supset -\frac{v}{\sqrt{2}} \Big[\bar{d}_L U_d M_d K^\dagger_d d_R + \bar{u}_L U_u M_u K^\dagger_u u_R \Big] + H.c.$$

→ So we can rotate $d_R \to K_d d_R$, $u_R \to K_u u_R$ and $d_L \to U_d d_L$ ($\Rightarrow \bar{d}_L \to \bar{d}_L U^\dagger_d$), $u_L \to U_u u_L$ to obtain (in the mass basis):

$$\mathcal{L}_Y^{(q)} \supset -\left[m_i^{(d)} \bar{d}_L^i d_R^i + m_i^{(u)} \bar{u}_L^i u_R^i \right] + H.c.$$

where $m_i^{(d)} = \frac{v}{\sqrt{2}} M_{d,ii}$, $m_i^{(u)} = \frac{v}{\sqrt{2}} M_{u,ii}$

and where $M_{d,ii}$ and $M_{u,ii}$ are the i-th eigenvalues of M_d and M_u, respectively.

- Now, how does this change of basis affect the quark-gauge field interactions?

↳ $D_\mu \Psi_j = (\partial_\mu - i g W_\mu^a \tau^a - i g' \frac{Y}{2} B_\mu) \Psi_j =$

$$= \left[\partial_\mu - \frac{i}{2} \begin{pmatrix} g W_\mu^3 + g' Y B_\mu & g(W_\mu^1 - i W_\mu^2) \\ g(W_\mu^1 + i W_\mu^2) & -g W_\mu^3 + g' Y B_\mu \end{pmatrix} \right] \Psi_j$$

↳ $i \sum_j \bar{\Psi}_j \not{D} \Psi_j \supset i \bar{Q}_L \not{D} Q_L + i \bar{u}_R \not{D} u_R + i \bar{d}_R \not{D} d_R =$

$$= (\bar{u}_L^i, \bar{d}_L^i) \left[i \not{\partial} + \gamma^\mu \begin{pmatrix} \frac{g}{2} W_\mu^3 + \frac{g'}{6} B_\mu & \frac{g}{2}(W_\mu^1 - i W_\mu^2) \\ \frac{g}{2}(W_\mu^1 + i W_\mu^2) & -\frac{g}{2} W_\mu^3 + \frac{g'}{6} B_\mu \end{pmatrix} \right] \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix} +$$

$$+ \bar{u}_R^i \left(i \not{\partial} + \frac{2}{3} g' \not{B} \right) u_R^i + \bar{d}_R^i \left(i \not{\partial} - \frac{1}{3} g' \not{B} \right) d_R^i$$

where we used $Q = T^3 + \frac{1}{2} Y_W \rightarrow Y(Q_L) = \frac{1}{3}$, $Y(u_R) = \frac{4}{3}$, $Y(d_R) = -\frac{2}{3}$

Q: electric charge

T^3: 3rd component of weak isospin

Y_W: weak hypercharge

↳ Note that in the original basis (flavour basis) the interaction terms don't mix generations.

↳ Furthermore, the hypercharge interactions don't mix flavours, so they are invariant under a change of basis $u_R \rightarrow K_u u_R$, $d_R \rightarrow K_d d_R$

↳ The same is true for the couplings to W_μ^3 and B_μ (since they come from the diagonal part).

↳ However, the interaction terms involving $W_\mu^\pm$ will lead to

$$\frac{g}{\sqrt{2}}\left[\bar{u}_L^i \gamma^\mu W_\mu^+ d_L^i + \bar{d}_L^i \gamma^\mu W_\mu^- u_L^i\right]$$

$$\xrightarrow[\text{mass basis}]{\text{going to}} \frac{g}{\sqrt{2}}\left[W_\mu^+ \bar{u}_L^i \gamma^\mu \underbrace{\left(U_u^\dagger U_d\right)}_{\equiv V}{}_{ij} d_L^j + W_\mu^- \bar{d}_L^i \gamma^\mu \underbrace{\left(U_d^\dagger U_u\right)}_{= V^\dagger}{}_{ij} u_L^j\right]$$

↳ V_{ij} is known as the <u>CKM-matrix</u>

(6) V is clearly unitary, so it can be written as $V = e^{iA}$, with $A^\dagger = A$.

↳ Since A is $N \times N$ Hermitian matrix, it has N^2 real d.o.f.s.

↳ If V were real, it would be an $N \times N$ orthogonal matrix and hence have $\frac{N}{2}(N-1)$ real d.o.f.s (number of independent rotations of $O(N)$).

↳ So V has $\frac{N}{2}(N-1)$ angles and $N^2 - \frac{N}{2}(N-1) = \frac{N}{2}(N+1)$ phases.

↳ Note that in the Yukawa sector there are $2N$ residual global $U(1)$ symmetries, since we can transform $d_L^j \to e^{i\alpha_j} d_L^j$, $d_R^j \to e^{i\alpha_j} d_R^j$ and $u_L^j \to e^{i\beta_j} u_L^j$, $u_R^j \to e^{i\beta_j} u_R^j$, with $j \in [1, \dots, N]$ leaving the mass terms invariant.

So we can use these quark field redefinitions to remove complex phases of V.

However, if all rotations are the same, $\alpha_j = \beta_j = \theta$, then V is unchanged, hence we can only remove $2N-1$ phases in this way.

↳ So there are $\frac{N}{2}(N+1) - (2N-1) = \frac{1}{2}\left(N^2 + N - 4N + 2\right) = \frac{1}{2}(N-1)(N-2)$ independent phases.

Therefore, for N quark generations, the CKM matrix has:

$$\boxed{\begin{array}{l} \frac{1}{2}N(N-1) \text{ mixing angles} \\ \frac{1}{2}(N-1)(N-2) \text{ complex phases} \end{array}}$$

$N = 2$: 1 angle (Cabibbo angle)

$N = 3$: 3 angles, 1 phase

(c) Let's look at the up-type quark mass terms

$$\mathcal{L}_Y^{(q)} \supset -\frac{v}{\sqrt{2}}\left[\underbrace{\bar{u}_L}_{=\bar{u}P_R}\Lambda_u \underbrace{u_R}_{=P_R u} + \underbrace{\bar{u}_R}_{=\bar{u}P_L}\Lambda_u^\dagger \underbrace{u_L}_{=P_L u}\right] = -\frac{v}{2\sqrt{2}}\left[\bar{u}\left(\Lambda_u+\Lambda_u^\dagger\right)u + \bar{u}\left(\Lambda_u-\Lambda_u^\dagger\right)\gamma^5 u\right]$$

where we used $P_{L/R} = \frac{1}{2}(1 \mp \gamma^5)$

↳ How does this transform under CP?

$u_i \xrightarrow{P} \gamma^0 u_i \quad , \quad u_i \xrightarrow{C} C\bar{u}_i^T$

Reminder: Discrete Symmetries of the Dirac Theory: Peskin, Schroeder - 3.6 (p. 64)

$\Rightarrow u_i \xrightarrow{CP} \gamma^0 C\bar{u}_i^T \quad \rightsquigarrow \bar{u}_i \xrightarrow{CP} u_i^T C\gamma^0$

$\Rightarrow \bar{u}_i u_j \xrightarrow{CP} \bar{u}_j u_i \quad , \quad \bar{u}_i\gamma^5 u_j \xrightarrow{CP} -\bar{u}_j\gamma^5 u_i$

↳ So the Yukawa term becomes

$$\xrightarrow{CP} -\frac{v}{2\sqrt{2}}\left[\bar{u}\left(\Lambda_u+\Lambda_u^\dagger\right)^T u - \bar{u}\left(\Lambda_u-\Lambda_u^\dagger\right)^T\gamma^5 u\right] =$$

$$= -\frac{v}{2\sqrt{2}}\left[\bar{u}\left(\Lambda_u^*+(\Lambda_u^\dagger)^*\right)u + \bar{u}\left(\Lambda_u^*-(\Lambda_u^\dagger)^*\right)\gamma^5 u\right]$$

↳ Note that $\bar{u}_i u_j$ and $\bar{u}_i\gamma^5 u_j$ are invariant under T (time reversal), so if the coefficients $(\Lambda_u^*)_{ij}$ etc. are complex, the above expression is not invariant under T, since $i \xrightarrow{T} -i$. But since CPT has to be a symmetry, this would imply that CP has to be violated.

↳ So the Yukawa sector of the quarks is only CP invariant, if $\Lambda_{u,d}^* = \Lambda_{u,d}$.

Note that this is not a basis-invariant statement. However, one can show that if the CKM matrix V is real, there is no CP-violation. So the experimental evidence of CP-violation implied that V cannot be real and hence the number of quark generations is $N > 2$.

(d) • For massless neutrinos there is no Yukawa term in the lepton sector like $-\Lambda_{ij}^{(\nu)}\bar{E}_L^i\tilde{H}\nu_R^j + H.c.$ (analogous to the up-like quark in the quark sector).

So in order to diagonalize $\Lambda^{(e)}_{ij} \bar{E}^i_L H e^j_R$, we just need two unitary matrices K_e and U_e to go to the mass basis via $e_R \to K_e e_R$, $e_L \to U_e e_L$.

- The lepton-gauge field interactions have the form

$$i\bar{E}_L \not{D} E_L + i\bar{e}_R \not{D} e_R = (\bar{\nu}^i_L, \bar{e}^i_L)\left[i\not{\partial} + \frac{1}{2}\gamma^\mu \begin{pmatrix} gW^3_\mu - g'B_\mu & gW^+_\mu \\ gW^-_\mu & -gW^3_\mu - g'B_\mu \end{pmatrix}\right]\begin{pmatrix} \nu^i_L \\ e^i_L \end{pmatrix} +$$

$$+ \bar{e}^i_R (i\not{\partial} - g'\not{B}) e_R$$

↳ Now let's look at the flavour mixing part

$$\frac{g}{2}\left[\bar{\nu}^i_L \not{W}^+ e^i_L + \bar{e}^i_L \not{W}^- \nu^i_L\right]$$

↳ Going to the mass-basis by $e_L \to U_e e_L$ can now be undone by the rotation $\nu_L \to U_e \nu_L$, which can be done with impunity, because there is no corresponding mass term in the Yukawa sector.

↳ Hence for massless neutrinos, the lepton sector can be diagonal in the mass terms and the interaction terms at the same time!

2 Higgs boson decay to fermions

The Yukawa interaction term is $\mathcal{L}_Y = \frac{g}{2}\frac{m_f}{M_W} h\bar{f}f = y h \bar{f} f$, where we introduced $y = \frac{g}{2}\frac{m_f}{M_W}$, to keep the expression short.

The (tree-level) amplitude reads

reminder: QED last semester, PS 11.

$$\mathcal{M} = y\, \bar{u}_{s_1}(\vec{p}_1)\, v_{s_2}(\vec{p}_2),$$

with $\left\{\begin{matrix}\vec{p}_1, s_1 \\ \vec{p}_2, s_2\end{matrix}\right\}$ the momenta & spins of the outgoing $\left\{\begin{matrix}\text{fermion} \\ \text{anti-fermion}\end{matrix}\right\}$.

Then,

$$\sum_{\text{spins}} |\mathcal{M}|^2 = y^2 \sum_{s_1, s_2} \bar{u}_{s_1}(\vec{p}_1) v_{s_2}(\vec{p}_2) \bar{v}_{s_2}(\vec{p}_2) u_{s_1}(\vec{p}_1) = y^2\, tr\left[(\not{p}_1 + m_f)(\not{p}_2 - m_f)\right] \underset{\uparrow\, tr[\gamma^\mu]=0}{=}$$

$$= y^2\, tr\left[\not{p}_1 \not{p}_2 - m_f^2\right] \underset{\uparrow\, tr[\gamma^\mu\gamma^\nu] = 4\eta^{\mu\nu}}{=} 4y^2(p_1 \cdot p_2 - m_f^2)$$

In the center-of-mass frame:

$$k^\mu = (m_h, \vec{0}) \;,\; p_1^\mu = \left(\frac{m_h}{2}, \vec{p}\right) \;,\; p_2^\mu = \left(\frac{m_h}{2}, -\vec{p}\right);$$

with 4-momentum conservation: $m_h = 2E_f$, $E_f^2 = \vec{p}^{\,2} + m_f^2$.

$$s = m_h^2 = (p_1+p_2)^2 = p_1^2 + p_2^2 + 2p_1\cdot p_2 = 2m_f^2 + 2p_1\cdot p_2 \Leftrightarrow p_1\cdot p_2 = \frac{m_h^2}{2} - m_f^2$$

Then,

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 4y^2\left(\frac{m_h^2}{2} - 2m_f^2\right) = 2y^2 m_h^2\left(1 - 4\frac{m_f^2}{m_h^2}\right)$$

The total decay rate is then given by

identical calculation to last semester QED, PS 11, Q. 1A2.

$$\Gamma = N_c \cdot \frac{1}{2E_{\vec{k}}} \sum_{\text{spins}} \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_{\vec{p}_1}} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_{\vec{p}_2}} (2\pi)^4 \delta^{(4)}(p_1+p_2-k)\,|\mathcal{M}|^2 =$$

$$= N_c \frac{y^2 m_h}{8\pi}\left(1 - 4\frac{m_f^2}{m_h^2}\right)^{3/2} = N_c\, m_h \frac{g^2 m_f^2}{32\pi M_W^2}\left(1 - 4\frac{m_f^2}{m_h^2}\right)^{3/2} =$$

$$= N_c \frac{G_F m_f^2 m_h}{4\sqrt{2}\,\pi}\left(1 - 4\frac{m_f^2}{m_h^2}\right)^{3/2}.$$

$$G_F = \frac{g^2}{4\sqrt{2}\,M_W^2}$$