Fermi theory: a low energy description of weak interaction

I: The charged sector of 4-Fermi theory

(a) The Lagrangian for the electroweak interactions in the Standard Model (without the Yukawa sector) is:

$$\mathcal{L}_{SM} \supset -\frac{1}{4} W^a_{\mu\nu} W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu H)^\dagger (D^\mu H) - \lambda \left(H^\dagger H - \frac{v^2}{2} \right)^2 +$$
$$+ i \sum_j \bar{\psi}_j \not{D} \psi_j$$

where
$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g \varepsilon^{abc} W^b_\mu W^c_\nu,$$
$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,$$
$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \quad H_{1,2} \in \mathbb{C},$$
$$D_\mu = \partial_\mu - i g W^a_\mu \tau^a - i g' \frac{Y}{2} B_\mu,$$
$$\psi_j = \{ Q_L, E_L, u_R, d_R, e_R \}, \quad Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L, \quad E_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L$$

That is, we have added $\boxed{\mathcal{L}_f = i \sum_j \bar{\psi}_j \not{D} \psi_j}$ to the $SU(2) \times U(1)$ theory from before.

(b) $\mathcal{L}_D = \bar{\psi} i \gamma_\mu \partial^\mu \psi$ ⇝ symmetries:
$$\psi \to \psi' = e^{i\alpha} \psi = \psi + \alpha \underbrace{i\psi}_{= \delta_V \psi}$$
$$\psi \to \psi' = e^{i\gamma_5 \beta} \psi = \psi + \beta \underbrace{i\gamma_5 \psi}_{= \delta_A \psi}$$

Noether current: $J^\mu = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi$ w/ φ any kind of field

$$\Rightarrow \boxed{J^\mu_V = i \bar{\psi} \gamma^\mu \delta_V \psi = -\bar{\psi} \gamma^\mu \psi} \qquad \text{Vector current}$$
$$\boxed{J^\mu_A = i \bar{\psi} \gamma^\mu \delta_A \psi = -\bar{\psi} \gamma^\mu \gamma_5 \psi} \qquad \text{Axial current}$$

Experiment: Wu et al '56

Parity is maximally violated in the β-decay, i.e. the weak interaction couples only to left handed fermions.

$$\Rightarrow \boxed{\begin{aligned} J^\mu_L &= \frac{1}{2}\left(J^\mu_V + J^\mu_A \right) = -\bar{\psi} \gamma^\mu \frac{1}{2}(1 + \gamma_5) \psi = \\ &= -\bar{\psi} \gamma^\mu P_L \psi = -\bar{\psi}_L \gamma^\mu \psi_L \end{aligned}} \qquad \text{„V-A theory"}$$

„V-A was the key", Steven Weinberg
(see „Weinberg S., Essay: Half a century of the SM, Phys. rev. lett. 2018" for more information)

<u>Note:</u> We work with $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ instead of $\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. In the latter case, we would have a relative minus between the currents, therefore justifying the name V−A. (Definitions in Weyl basis).

(c) $\mathcal{L}_{SM} \supset \bar{Q}_L i\not{D} Q_L + \bar{E}_L i\not{D} E_L$ w/ $Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L$, $E_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L$

Consider the first term (second term is analogous) in the following.

$$\bar{Q}_L i\not{D} Q_L = \bar{Q}_L i\gamma^\mu \big(\partial_\mu \underbrace{- ig W_\mu^a \tau^a - ig' \frac{Y(Q_L)}{2} B_\mu}\big) Q_L$$

$$= -i \begin{pmatrix} \frac{1}{2}(g' Y(Q_L) B_\mu + g W_\mu^3) & \frac{1}{2}(g W_\mu^1 - ig W_\mu^2) \\ \frac{1}{2}(g W_\mu^1 + ig W_\mu^2) & \frac{1}{2}(g' Y(Q_L) B_\mu - g W_\mu^3) \end{pmatrix} =$$

$$\xrightarrow{W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2)} = -i \begin{pmatrix} \cdots & \frac{g}{\sqrt{2}} W_\mu^+ \\ \frac{g}{\sqrt{2}} W_\mu^- & \cdots \end{pmatrix}$$

$$\Rightarrow \bar{Q}_L i\not{D} Q_L = \frac{g}{\sqrt{2}} W_\mu^+ (\bar{u}_L \gamma^\mu d_L) + \frac{g}{\sqrt{2}} W_\mu^- (\bar{d}_L \gamma^\mu u_L) + \text{diagonal part}$$

Analogous for the leptons. In total, i.e. for the sum of the two, we find

$$\boxed{\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} (W_\mu^+ J^{\mu +} + W_\mu^- J^{\mu -}) \text{ w/ } J^{\mu +} = \bar{u}_L \gamma^\mu d_L + \bar{\nu}_L \gamma^\mu e_L, \quad J^{\mu -} = \bar{d}_L \gamma^\mu u_L + \bar{e}_L \gamma^\mu \nu_e}$$

Since the currents are purely left-handed, they respect the prescription of (b). Note that under Hermitian conjugation $(W_\mu^+)^\dagger = W_\mu^-$ & $(J_\mu^+)^\dagger = J_\mu^-$.

<u>Note:</u> In the literature you may find J_μ^+ & J_μ^- defined w/ opposite signs, i.e.

$$J^{\mu +} = \bar{d}_L \gamma^\mu u_L + \bar{e}_L \gamma^\mu \nu_e$$

$$J^{\mu -} = \bar{u}_L \gamma^\mu d_L + \bar{\nu}_L \gamma^\mu e_L$$

(d) We need to express $\mathcal{L}$ in terms of $W^{\pm}$.

Mass term From the previous problem set we know that after SSB the mass term is

$$m_W^2 W_\mu^+ W^{\mu -} \quad w/ \quad m_W = \frac{gv}{2}$$

Kinetic term

- $W_\mu^{\pm} = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp i W_\mu^2\right)$

$$\Rightarrow W_\mu^1 = \frac{1}{\sqrt{2}}\left(W_\mu^+ + W_\mu^-\right)$$

$$W_\mu^2 = \frac{i}{\sqrt{2}}\left(W_\mu^+ - W_\mu^-\right)$$

Consider only Abelian part of field strength $W_{\mu\nu}^a$ here, the non-Abelian part will play no role in the low energy theory:

- $(W_{\mu\nu}^1)^2 = (\partial_\mu W_\nu^1 - \partial_\nu W_\mu^1)^2 = \frac{1}{2}\left(\partial_\mu W_\nu^+ + \partial_\mu W_\nu^- - \partial_\nu W_\mu^+ - \partial_\nu W_\mu^-\right)^2 =$
$$= \frac{1}{2}\left(W_{\mu\nu}^+ + W_{\mu\nu}^-\right)^2$$

- $(W_{\mu\nu}^2)^2 = (\partial_\mu W_\nu^2 - \partial_\nu W_\mu^2)^2 = -\frac{1}{2}\left(\partial_\mu W_\nu^+ - \partial_\mu W_\nu^- - \partial_\nu W_\mu^+ + \partial_\nu W_\mu^-\right)^2 =$
$$= -\frac{1}{2}\left(W_{\mu\nu}^+ - W_{\mu\nu}^-\right)^2$$

$$\Rightarrow \mathcal{L} \supset -\frac{1}{4} W_{\mu\nu}^a W^{\mu\nu a} \supset -\frac{1}{4}(W_{\mu\nu}^1)^2 - \frac{1}{4}(W_{\mu\nu}^2)^2 =$$

$$= -\frac{1}{8}\left(\underline{(W_{\mu\nu}^+)^2} + \underline{(W_{\mu\nu}^-)^2} + 2W_{\mu\nu}^+ W^{\mu\nu -}\right) + \frac{1}{8}\left(\underline{(W_{\mu\nu}^+)^2} + \underline{(W_{\mu\nu}^-)^2} - 2W_{\mu\nu}^+ W^{\mu\nu -}\right) =$$

$$= -\frac{1}{2} W_{\mu\nu}^+ W^{\mu\nu -}$$

In total, neglecting higher order interactions, we obtain:

$$\boxed{\mathcal{L}[W^{\pm}] = -\frac{1}{2} W_{\mu\nu}^+ W^{\mu\nu -} + m_W^2 W_\mu^+ W^{\mu -} + \frac{g}{\sqrt{2}}\left(W_\mu^+ J^{\mu +} + W_\mu^- J^{\mu -}\right)}$$

EOM: $\partial_\mu W^{\mu\nu -} + m_W^2 W^{\nu -} = -\frac{g}{\sqrt{2}} J^{\nu +}$

& $m_W^2 \partial_\nu W^{\nu -} = -\frac{g}{\sqrt{2}} \partial_\nu J^{\nu +} \underset{\substack{\uparrow \\ \text{conserved} \\ \text{current}}}{=} 0$ as constraint

$$\xrightarrow{\text{impose constraint}} \boxed{(\Box + m_W^2) W_\nu^- = -\frac{g}{\sqrt{2}} J_\nu^+}$$

Solution: $W^{\mu -} = -\frac{g}{\sqrt{2}} \frac{J^{\mu +}}{\Box + m_W^2} \underset{\substack{\text{low energy} \\ E \ll m_W}}{\simeq} -\frac{g}{\sqrt{2} m_W^2} J^{\mu +}$ & similarly for $W^{\mu +}$.

Insert back into $\mathcal{L}$:

$$\boxed{\begin{aligned}\mathcal{L}_{\substack{\text{low}\\ \text{energy}}} &= -\frac{g^2}{2m_W^2} J_\mu^+ J^{\mu -} = -2\sqrt{2}\, G_F\, J_\mu^+ J^{\mu -} + \mathcal{O}\left(\frac{1}{m_W^4}\right)\\ \text{w/}\quad G_F &= \frac{g^2}{4\sqrt{2}\, m_W^2}\end{aligned}}$$

This is the lagrangian of the famous 4-Fermi theory. (The numerical factors in the definition of G_F come from the comparison with the original Fermi-Lagrangian and thus are of historic origin.)

(e) $[G_F] = -2 \;\to$ <u>not</u> renormalizable

Reminder: [see e.g. Peskin-Schroeder, 10.1]

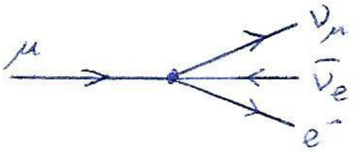
A theory w/ a coupling constant α is:

Super-Renormalizable : $[\alpha] > 0$

Renormalizable : $[\alpha] = 0$

Non-Renormalizable : $[\alpha] < 0$

(f) <u>$\mu \to e^- + \nu_\mu + \bar{\nu}_e$</u>



$\sim G_F E^2$

$$\Rightarrow \Gamma \sim |\mathcal{M}|^2 E \sim G_F^2 E^5 \sim \left(\frac{g^2}{m_W^2}\right)^2 E^5 \overset{m_W \sim g v}{\sim} \frac{E^5}{v^4}$$

The energy scale in the β-decay of a muon is given by

$E \sim m_\mu - m_e \simeq m_\mu$

$$\Rightarrow \Gamma \sim \frac{m_\mu^5}{v^4} = \frac{1}{\tau_\mu} \iff \boxed{v \sim \left(m_\mu^5 \cdot \tau_\mu\right)^{1/4}}$$

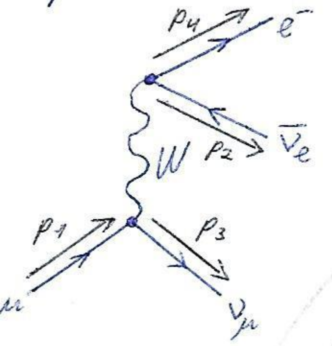
Inserting $m_\mu \sim 10^2\,\text{MeV} = 10^{-1}\,\text{GeV}$,

$\tau_\mu \sim 10^{-6}\,\text{s} \sim 10^{-6} \cdot \frac{10^{15}}{\text{eV}} \sim \frac{10^{18}}{\text{GeV}}$, we find

$$v \sim \left(10^{-5}\,\text{GeV}^5 \cdot 10^{18}\,\text{GeV}^{-1}\right)^{1/4} \sim \left(10^{13}\,\text{GeV}^4\right)^{1/4} \sim 10^3\,\text{GeV}$$ <u>close enough!</u>

[Exact values: $\Gamma \simeq \frac{G_F^2 m_\mu^5}{192\pi^3}$, $v \simeq 246\,\text{GeV}$, see Griffiths - Introduction to Elementary Particles; 9.2]

Explicit calculation:



$$\mathcal{M} = 2\sqrt{2}\,G_F\left[\bar{u}_3 \gamma^\mu P_L u_1\right]\left[\bar{u}_4 \gamma_\mu P_L v_2\right]$$

$$\mathcal{M}^\dagger = 2\sqrt{2}\,G_F\left[u_1^\dagger P_L^\dagger \gamma^{\mu\dagger} \gamma_0^\dagger u_3\right]\left[v_2^\dagger P_L^\dagger \gamma_\mu^\dagger \gamma_0^\dagger u_4\right] =$$

$$= 2\sqrt{2}\,G_F\left[\bar{u}_1 \gamma_0 P_L \gamma_0 \gamma^\mu u_3\right]\left[\bar{v}_2 \gamma_0 P_L \gamma_0 \gamma_\mu u_4\right] =$$

$$= 2\sqrt{2}\,G_F\left[\bar{u}_1 \underbrace{\gamma_0^2}_{=1} P_R \gamma^\mu u_3\right]\left[\bar{v}_2 \underbrace{\gamma_0^2}_{=1} P_R \gamma_\mu u_4\right]$$

$\{\gamma_5, \gamma^\mu\} = 0$
$\Rightarrow P_L \gamma_\mu = \gamma_\mu P_R$

reminder:
$\bar{\psi} = \psi^\dagger \gamma_0$
$\gamma_0^\dagger = \gamma_0$
$\gamma_0^2 = 1$
$P_L^\dagger = P_L$
$(\gamma^\mu)^\dagger = \gamma_0 \gamma^\mu \gamma_0$

$$\Rightarrow |\mathcal{M}|^2 = (2\sqrt{2}\,G_F)^2 \left[\bar{u}_3 \gamma^\mu P_L u_1\right]\left[\bar{u}_1 P_R \gamma^\nu u_3\right]\left[\bar{u}_4 \gamma_\mu P_L v_2\right]\left[\bar{v}_2 P_R \gamma_\nu u_4\right]$$

Sum over spins, average over initial ones.
The muon has two possible polarizations $\rightarrow \frac{1}{2}$.

$$\Rightarrow \langle|\mathcal{M}|^2\rangle = \frac{1}{2}(2\sqrt{2}\,G_F)^2 \sum_{\substack{\text{all}\\\text{spins}}} \bar{u}_3^{\,i} (\gamma^\mu P_L)_{ij} u_1^{\,j} \bar{u}_1^{\,k} (P_R \gamma^\nu)_{kl} u_3^{\,l} \cdot$$

$$\cdot \bar{u}_4^{\,a} (\gamma_\mu P_L)_{ab} v_2^{\,b} \bar{v}_2^{\,c} (P_R \gamma_\nu)_{cd} u_4^{\,d} =$$

Casimir trick

$$\begin{cases} P_L P_R = 0 \\ P_L \gamma_\mu = \gamma_\mu P_R \\ P_L^2 = P_L \end{cases}$$

$$= \frac{1}{2}\cdot 8 G_F^2 \, tr\left[\not{p}_3 \gamma^\mu P_L (\not{p}_1 - m_\mu) P_R \gamma^\nu\right] tr\left[\not{p}_4 \gamma_\mu P_L \not{p}_2 P_R \gamma_\nu\right] =$$

$$= 4 G_F^2 \, tr\left[\not{p}_3 \gamma^\mu P_L \not{p}_1 \gamma^\nu\right] tr\left[\not{p}_4 \gamma_\mu P_L \not{p}_2 \gamma_\nu\right] =$$

$$= \cancel{\frac{4}{4}} G_F^2 \, tr\left[\not{p}_3 \gamma^\mu (1+\gamma_5) \not{p}_1 \gamma^\nu\right] tr\left[\not{p}_4 \gamma_\mu (1+\gamma_5) \not{p}_2 \gamma_\nu\right]$$

1st trace: $p_{3\alpha} p_{1\beta}\, tr\left[\gamma^\alpha \gamma^\mu (1+\gamma_5) \gamma^\beta \gamma^\nu\right] =$

$$= p_{3\alpha} p_{1\beta}\left[4\left(\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\alpha\nu}\eta^{\beta\mu}\right) + 4i\,\varepsilon^{\alpha\mu\beta\nu}\right] =$$

$$= 4\left(p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - (p_3 \cdot p_1)\eta^{\mu\nu} + i\,p_{3\alpha} p_{1\beta}\,\varepsilon^{\alpha\mu\beta\nu}\right)$$

2nd trace: re-label $\begin{smallmatrix} 3 \rightarrow 4 \\ 1 \rightarrow 2 \end{smallmatrix}$

$$= 4\left(p_4^\mu p_2^\nu + p_4^\nu p_2^\mu - (p_4 \cdot p_2)\eta^{\mu\nu} + i\,p_{4\tilde{\alpha}} p_{2\tilde{\beta}}\,\varepsilon^{\tilde{\alpha}\mu\tilde{\beta}\nu}\right)$$

$$\Rightarrow \langle|\mathcal{M}|^2\rangle = 16 G_F^2 \left(p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - (p_3 \cdot p_1)\eta^{\mu\nu} + i\,p_{3\alpha} p_{1\beta}\,\varepsilon^{\alpha\mu\beta\nu}\right) \cdot$$

$$\cdot \left(p_{4\mu} p_{2\nu} + p_{4\nu} p_{2\mu} - (p_4 \cdot p_2)\eta_{\mu\nu} + i\,p_{4\tilde{\alpha}} p_{2\tilde{\beta}}\,\varepsilon^{\tilde{\alpha}}{}_{\mu}{}^{\tilde{\beta}}{}_{\nu}\right)$$

$$\langle|\mathcal{M}|^2\rangle = 16\, G_F^2 \Big[(p_3\cdot p_4)(p_1\cdot p_2) + (p_3\cdot p_2)(p_1\cdot p_4) \underline{- (p_3\cdot p_1)(p_4\cdot p_2)} +$$

$$+ (p_3\cdot p_2)(p_1\cdot p_4) + (p_3\cdot p_4)(p_1\cdot p_2) \underline{- (p_3\cdot p_1)(p_4\cdot p_2)} -$$

$$\underline{- (p_3\cdot p_1)(p_4\cdot p_2)} \underline{- (p_3\cdot p_1)(p_4\cdot p_2)} + \underline{4 (p_3\cdot p_1)(p_4\cdot p_2)} +$$

only $\varepsilon\varepsilon$ contraction not vanishing; other terms are symmetric in $\mu\nu$.

$$+ (-1)\, p_{3\alpha}\, p_{1\beta}\, p_{4\tilde{\alpha}}\, p_{2\tilde{\beta}} \underbrace{\varepsilon^{\alpha\mu\beta\nu}\, \varepsilon^{\tilde{\alpha}}{}_{\mu}{}^{\tilde{\beta}}{}_{\nu}}_{= -2\left(\eta^{\alpha\tilde{\alpha}}\eta^{\beta\tilde{\beta}} - \eta^{\alpha\tilde{\beta}}\eta^{\beta\tilde{\alpha}}\right)} \Big] =$$

$$= 16\, G_F^2 \Big[2 (p_3\cdot p_4)(p_1\cdot p_2) + \underline{2 (p_3\cdot p_2)(p_1\cdot p_4)} +$$

$$+ 2 (p_3\cdot p_4)(p_1\cdot p_2) \underline{- 2 (p_3\cdot p_2)(p_1\cdot p_4)} \Big] =$$

$$= 64\, G_F^2 (p_1\cdot p_2)(p_3\cdot p_4)$$

Go to rest-frame of muon: $p_1 = \begin{pmatrix} m_\mu \\ 0 \end{pmatrix}$

$\rightarrow (p_1\cdot p_2) = m_\mu E_2$

$\rightarrow$ since $p_1 = p_2 + p_3 + p_4$, $(p_3+p_4)^2 = p_3^2 + p_4^2 + 2 p_3\cdot p_4 = m_e^2 + 2 p_3\cdot p_4 =$

$$= (p_1 - p_2)^2 = p_1^2 + p_2^2 - 2 p_1\cdot p_2 = m_\mu^2 - 2 p_1\cdot p_2$$

$$\Rightarrow p_3\cdot p_4 = \frac{m_\mu^2 - m_e^2}{2} - m_\mu E_2 \simeq \frac{m_\mu}{2}(m_\mu - 2E_2)$$

$$\Rightarrow \langle|\mathcal{M}|^2\rangle = 32\, G_F^2 m_\mu^2 E_2 (m_\mu - 2E_2) = 32\, G_F^2 m_\mu^2 |\vec{p}_2| (m_\mu - 2|\vec{p}_2|)$$

<u>Decay rate:</u> $1 \rightarrow 2 + \ldots + n$

$$\Gamma = \frac{S}{2M} \int \langle|\mathcal{M}|^2\rangle (2\pi)^4 \delta^{(4)}(p_1 - p_2 - \ldots - p_n) \prod_{j=2}^{n} dLIPS_j$$

w/ S: combinatorial factor for identical particles, i.e. for each identical particle of multiplicity n we divide by $n!$

M: mass of decaying particle

$$dLIPS_j = \frac{d^3\vec{p}_j}{(2\pi)^3\, 2E_j}$$

<u>Our case:</u>

$$\Gamma = \frac{1}{2m_\mu} \int \overbrace{\langle|\mathcal{M}|^2\rangle}^{\text{fct. of } E_2} (2\pi)^4 \delta^{(4)}(p_1 - p_2 - p_3 - p_4) \frac{d^3\vec{p}_2}{(2\pi)^3 2|\vec{p}_2|} \frac{d^3\vec{p}_3}{(2\pi)^3 2|\vec{p}_3|} \frac{d^3\vec{p}_4}{(2\pi)^3 2|\vec{p}_4|}$$

$$\rightarrow \delta^{(4)}(p_1 - p_2 - p_3 - p_4) = \delta(m_\mu - |\vec{p}_2| - |\vec{p}_3| - |\vec{p}_4|)\, \delta^{(3)}(\vec{p}_2 + \vec{p}_3 + \vec{p}_4)$$

Perform $\vec{p}_3$ integral w/ $\delta^{(3)}(\ldots)$

$$\Rightarrow \Gamma = \frac{1}{2m_\mu} \int \frac{\langle|\mathcal{M}|^2\rangle}{8(2\pi)^5} \delta(m_\mu - |\vec{p}_2| - |\vec{p}_2 + \vec{p}_4| - |\vec{p}_4|) \frac{d^3\vec{p}_2\, d^3\vec{p}_4}{|\vec{p}_2||\vec{p}_2 + \vec{p}_4||\vec{p}_4|}$$

$\Rightarrow$ Do $\vec{p_2}$ integral. Go to spherical coordinates in mom. space w/ polar axis along $\vec{p_4}$. We have

$$d^3p_2 = |\vec{p_2}|^2 \, d|\vec{p_2}| \sin\theta \, d\theta \, d\varphi$$

$$|\vec{p_2}+\vec{p_4}|^2 = |\vec{p_2}|^2 + |\vec{p_4}|^2 + 2|\vec{p_2}||\vec{p_4}|\cos\theta \equiv u^2$$

$$\Gamma = \frac{1}{2m_\mu}\int \frac{\langle|\mathcal{M}|^2\rangle}{8(2\pi)^5}|\vec{p_2}|^2 \frac{d|\vec{p_2}|\, d^3\vec{p_4}}{|\vec{p_2}||\vec{p_4}|} \underbrace{\int_0^{2\pi} d\varphi}_{=2\pi} \underbrace{\int_0^{\pi} d\theta \sin\theta \, \frac{\delta(m_\mu - |\vec{p_2}| - |\vec{p_2}+\vec{p_4}| - |\vec{p_4}|)}{u(\theta)}}_{\text{go to } u \text{ integration}}$$

$$u^2 = |\vec{p_2}|^2 + |\vec{p_4}|^2 + 2|\vec{p_2}||\vec{p_4}|\cos\theta \quad \Big| \, d$$

$$\Rightarrow 2u\,du = -2|\vec{p_2}||\vec{p_4}|\sin\theta\, d\theta \quad \Leftrightarrow \quad \sin\theta\, d\theta = -\frac{u\,du}{|\vec{p_2}||\vec{p_4}|}$$

$$\Rightarrow \Gamma = \int \frac{\langle|\mathcal{M}|^2\rangle}{16 m_\mu (2\pi)^4}\frac{d^3\vec{p_4}\, d|\vec{p_2}|}{|\vec{p_4}|^2}\int_{u_-}^{u_+} du\, \delta(m_\mu - |\vec{p_2}| - |\vec{p_4}| - u)$$

$$\text{w/ } u_\pm \equiv \sqrt{|\vec{p_2}|^2 + |\vec{p_4}|^2 \pm 2|\vec{p_2}||\vec{p_4}|} = \big|\,|\vec{p_2}| \pm |\vec{p_4}|\,\big|$$

(We used the $-$ to change integration domain, i.e. $-\int_{u_+}^{u_-} = \int_{u_-}^{u_+}$)

For du integral, we must have $u_- < m_\mu - |\vec{p_2}| - |\vec{p_4}| < u_+$

$$\Leftrightarrow \begin{cases} |\vec{p_2}| < \frac{m_\mu}{2} & \text{: lower bound} \\ |\vec{p_4}| < \frac{m_\mu}{2} & \\ |\vec{p_2}| + |\vec{p_4}| > \frac{m_\mu}{2} & \text{: upper bound} \end{cases}$$

limits $|\vec{p_2}|$ & $|\vec{p_4}|$ integration: $|\vec{p_2}|$ runs from $\frac{m_\mu}{2} - |\vec{p_4}|$ to $\frac{m_\mu}{2}$, $|\vec{p_4}|$ runs from 0 to $\frac{m_\mu}{2}$

$$\Rightarrow \Gamma = \int \frac{32 G_F^2 m_\mu^2}{16 m_\mu (2\pi)^4}\frac{d^3\vec{p_4}}{|\vec{p_4}|^2}\int_{\frac{m_\mu}{2}-|\vec{p_4}|}^{\frac{m_\mu}{2}} d|\vec{p_2}|\; |\vec{p_2}|(m_\mu - 2|\vec{p_2}|) =$$

$$= \frac{2G_F^2 m_\mu}{(2\pi)^4}\int \frac{d^3\vec{p_4}}{|\vec{p_4}|^2}\left[|\vec{p_2}|^2\left(\frac{m_\mu}{2} - \frac{2}{3}|\vec{p_2}|\right)\right]_{\frac{m_\mu}{2}-|\vec{p_4}|}^{\frac{m_\mu}{2}} =$$

$$= \frac{2G_F^2 m_\mu}{(2\pi)^4}\int \frac{d^3\vec{p_4}}{|\vec{p_4}|^2}|\vec{p_4}|^2\left(\frac{m_\mu}{2} - \frac{2}{3}|\vec{p_4}|\right) = \frac{G_F^2 m_\mu^2}{(2\pi)^4}\int d^3\vec{p_4}\left(1 - \frac{4|\vec{p_4}|}{3m_\mu}\right)$$

Finally, write $d^3\vec{p_4} = 4\pi|\vec{p_4}|^2 d|\vec{p_4}| = 4\pi E^2 dE$, w/ $E \mathrel{\hat=}$ energy of the e^-

$$\Rightarrow \boxed{\frac{d\Gamma}{dE} = \frac{G_F^2 m_\mu^2 E^2}{4\pi^3}\left(1 - \frac{4E}{3m_\mu}\right)}$$

Energy distribution of emitted electron! (nicely matches experiments)

Total decay rate:

$$\Gamma = \frac{G_F^2 m_\mu^2}{4\pi^3} \underbrace{\int_0^{\frac{m_\mu}{2}} dE\, E^2 \left(1 - \frac{4E}{3m_\mu}\right)}$$

$$= \left[\frac{E^3}{3}\left(1 - \frac{E}{m_\mu}\right)\right]_0^{\frac{m_\mu}{2}} = \frac{m_\mu^3}{3\cdot 8}\left(1 - \frac{1}{2}\right) = \frac{m_\mu^3}{3\cdot 16}$$

$$\boxed{\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}}$$

or, equivalently, $\tau_\mu = \frac{1}{\Gamma} = \frac{192\pi^3}{G_F^2 m_\mu^5} = \frac{384\pi^3 v^4}{m_\mu^5}$

$$G_F = \frac{g^2}{4\sqrt{2}\, m_W^2} = \frac{1}{\sqrt{2}\, v^2}$$

$$m_W = \frac{g v}{2}$$

Therefore, $v = \left(\frac{m_\mu^5 \tau_\mu}{384\pi^3}\right)^{1/4} =$

$$\simeq \left(\frac{(105{,}7\cdot 10^{-3}\,\mathrm{GeV})^5 \cdot 2{,}197\cdot 10^{-6}\cdot \frac{10^{25}}{6{,}582}\,\mathrm{GeV}^{-1}}{384\pi^3}\right)^{1/4} \simeq$$

$$\simeq 246{,}6\ \mathrm{GeV}$$ very close!

In fact, $v = 246{,}2196(2)$ GeV.

Note: since all masses in the SM are $\sim v \times$ coupling constant, the requirement of weak coupling gives an upper bound for all masses!