MPS I: Matrix product states

1. Overlaps and normalization $\langle\tilde\psi|\psi\rangle$

Consider overlap of 2-site MPS:

dummy index

$$|\beta\rangle = |\sigma_2\rangle|\sigma_1\rangle A^{1\sigma_1}{}_{\alpha} B^{\alpha\sigma_2}{}_{\beta} \qquad (1)$$

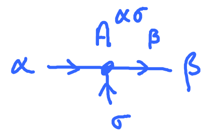
$$\langle\beta'|\beta\rangle = \underbrace{\overline{A^{1\sigma_1'}{}_{\alpha'}}}_{\parallel}\;\underbrace{\overline{B^{\alpha'\sigma_2'}{}_{\beta'}}}_{\parallel}\;\langle\sigma_1'|\underbrace{\langle\sigma_2'|\sigma_2\rangle}_{\delta^{\sigma_2'}{}_{\sigma_2}}|\sigma_1\rangle_{\delta^{\sigma_1'}{}_{\sigma_1}}\; A^{1\sigma_1}{}_{\alpha} B^{\alpha\sigma_2}{}_{\beta} \qquad (2)$$

introduce $A^{\dagger}$'s: $\qquad A^{\dagger\alpha'}{}_{\sigma_1' 1} \qquad B^{\dagger\beta'}{}_{\sigma_2'\alpha'} \qquad (3)$

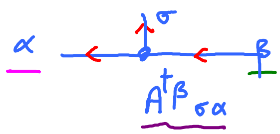
reorder $A^{\dagger}$'s: $\qquad B^{\dagger\beta'}{}_{\sigma_2'\alpha'}\,A^{\dagger\alpha'}{}_{\sigma_1'1}\,A^{1\sigma_1}{}_{\alpha}\,B^{\alpha\sigma_2}{}_{\beta} \qquad (4)$

Use diagrammatic rules to keep track of contraction patterns:

Ket: $\qquad |\beta\rangle = |\sigma\rangle|\alpha\rangle A^{\alpha\sigma}{}_{\beta} \qquad (5)$



Bra: $\qquad \langle\beta| = \langle\alpha|\langle\sigma|\,\overline{A^{\alpha\sigma}{}_{\beta}} \equiv A^{\dagger\beta}{}_{\sigma\alpha}\,\langle\alpha|\langle\sigma| \qquad (6)$



We accommodated complex conjugation via Hermitian conjugation and index transposition:

$$A^{\dagger\beta}{}_{\sigma\alpha} = \overline{A^{\alpha\sigma}{}_{\beta}}$$

This moves upstairs indices downstairs and vice versa, i.e. invents all arrows in diagram.
Note that in diagram vertex, α sits left, β right, whereas on $A^{\dagger}$, β sits left, α right.
This convention may seem initially awkward, but it greatly simplifies the structure of diagrams representing overlaps.

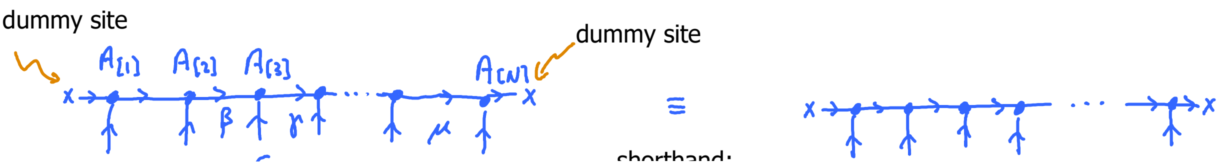
Generalization to many-site MPS:

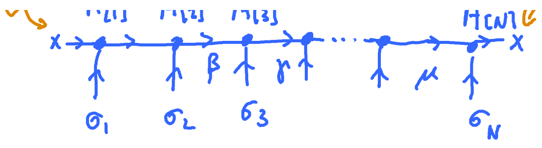
dummy index

$$|\psi\rangle = \overbrace{|\vec\sigma\rangle_N}^{|\sigma_N\rangle\dots|\sigma_2\rangle|\sigma_1\rangle} A^{1\sigma_1}_{[1]\,\alpha}\, A^{\alpha\sigma_2}_{[2]\,\beta}\, A^{\beta\sigma_3}_{[3]\,\gamma} \cdots A^{\mu\sigma_N}_{[N]\,1} \qquad (7)$$

dummy index

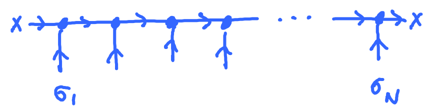
Square brackets indicate that each site has a different A matrix. We will often omit them and use the shorthand, $A^{\alpha\sigma_\ell}{}_{\beta} \equiv A^{\alpha\sigma_\ell}_{[\ell]\,\beta}$, since the ℓ on σ_ℓ uniquely identifies the site.





$\equiv$

shorthand: omit indices

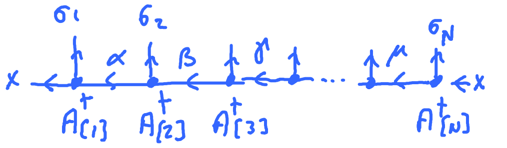


(8)

Recipe for ket formula: as chain grows, attach new matrices on the right (in <u>same</u> order as vertices in diagram); resulting in a matrix product structure.

Bra: $\langle\sigma_1|\langle\sigma_2|\ldots\langle\sigma_N|$

$$\langle\psi| = {}_N\langle\vec{\sigma}|\,\overline{A^{1\sigma_1}_{[1]\alpha}}\,\overline{A^{\alpha\sigma_2}_{[2]\beta}}\,\overline{A^{\beta\sigma_3}_{[3]\gamma}}\cdots\overline{A^{\mu\sigma_N}_{[N]1}} = A^{\dagger 1}_{[N]\sigma_N\mu}\cdots A^{\dagger\gamma}_{[3]\sigma_3\beta}\,A^{\dagger\beta}_{[2]\sigma_2\alpha}\,A^{\dagger\alpha}_{[1]\sigma_1 1}\,{}_N\langle\vec{\sigma}| \qquad (9)$$



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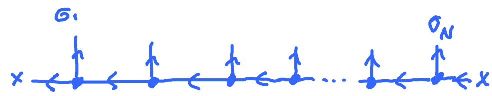
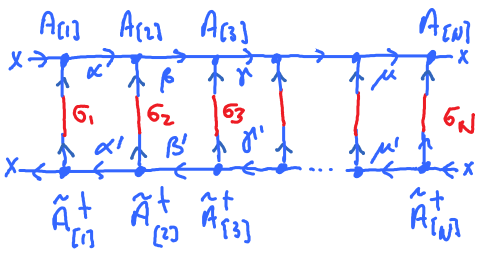


We expressed all matrices via their Hermitian conjugates by transposing indices and inverting arrows. To recover a matrix product structure, we ordered the Hermitian conjugate matrices to appear in the <u>opposite</u> order as the vertices in the diagram.

Recipe for bra formula: as chain grows, attach new matrices $A^\dagger$ on the left, opposite to vertex order in diagram.

Now consider overlap between two MPS:

$\langle\tilde{\psi}|\psi\rangle =$



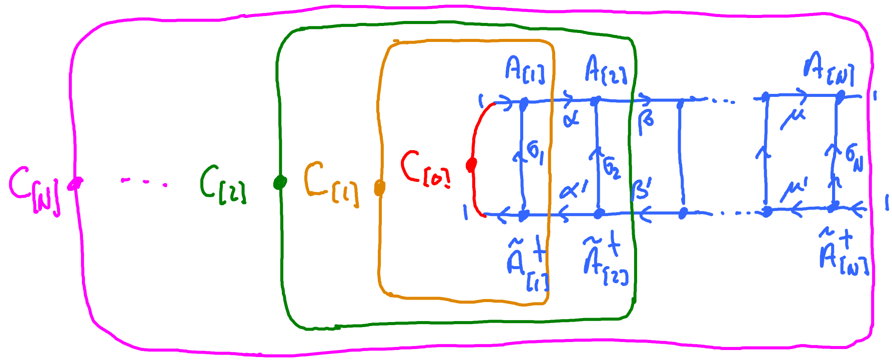
Recipe: contract all physical indices!

$$= \tilde{A}^{\dagger 1}_{[N]\sigma_N\mu'}\cdots\tilde{A}^{\dagger\mu'}_{[3]\beta'}\,\tilde{A}^{\beta'}_{[2]\sigma_2\alpha'}\,\tilde{A}^{\alpha'}_{[1]\sigma_1 1}\,A^{1\sigma_1}_{[1]\alpha}\,A^{\alpha\sigma_2}_{[2]\beta}\,A^{\beta}_{[3]\gamma}\cdots A^{\mu\sigma_N}_{[N]1} \qquad (10)$$

Exercise: derive this result algebraically from (7), (9),

If we would perform the matrix multiplication first, for fixed $\vec{\sigma}$, and then sum over $\vec{\sigma}$, we would get d^N terms, each of which is a product of $2N$ matrices. Exponentially costly! ☹

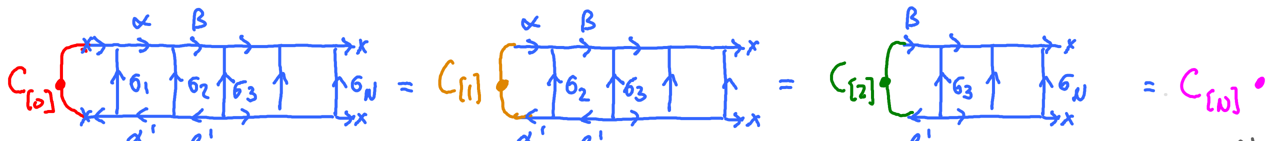
But calculation becomes tractable if we rearrange summations:

$\langle\tilde\psi|\psi\rangle =$ 

(11)

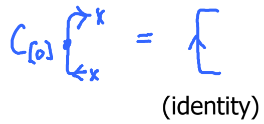
$$= \underbrace{\tilde A^{\dagger 1}{}_{\sigma_N\mu'} \cdots \underbrace{\tilde A^{\dagger\beta'}{}_{\sigma_2\alpha'} \underbrace{A^{\dagger\alpha'}{}_{\sigma_1 1}\cdot \overset{\equiv C'_{[0]1}}{1} \cdot A^{1\sigma_1}_{[1]\,\alpha}}_{\equiv C^{\alpha'}_{[1]\alpha}} A^{\alpha\sigma_2}_{[2]\,\beta}}_{\equiv C^{\beta'}_{[2]\beta}} \cdots A^{\mu\sigma_N}_{[N]\,1}}_{\equiv C'_{[N]1}} \qquad (12)$$

Diagrammatic depiction: 'closing zipper' from left to right.

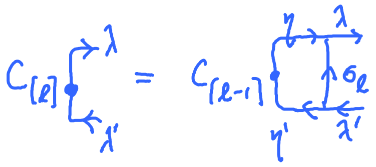


(13)

The set of two-leg tensors $C_{[\ell]}$ can be computed iteratively:

Initialization: 

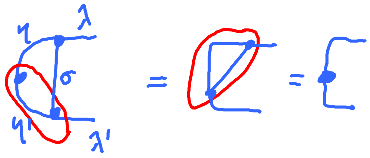
$C'_{[0]1} = 1$ (14)

Iteration step: 

$$C_{[\ell]}{}^{\lambda'}{}_{\lambda} = \tilde A^{\dagger\lambda'}{}_{\sigma_\ell\eta'}\, C^{\eta'}_{[\ell-1]\eta}\, A^{\eta\sigma_\ell}{}_{\lambda} \qquad (15)$$

Final answer: $\langle\tilde\psi|\psi\rangle = C'_{[N]1}$ (16)

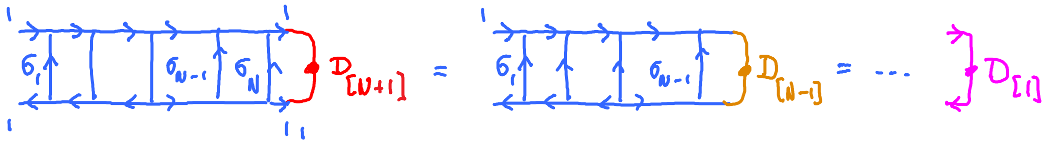
Cost estimate (if all A's are $D\times D$):

One iteration: $\underbrace{D^2 d}_{\substack{\text{fixed}\\ \lambda'\eta\sigma}} \cdot \underbrace{D}_{\substack{\text{sum}\\ \eta'}} + \underbrace{D^2}_{\substack{\text{fixed}\\ \lambda'\lambda}}\ \underbrace{dD}_{\substack{\text{sum}\\ \eta\sigma}}$ 

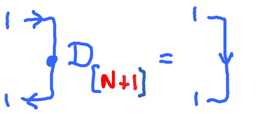
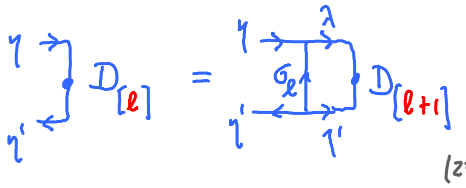
(17)

Total cost: $\sim D^3 d \cdot N$ (18)

Remark: a similar iteration scheme can be used to 'close zipper from right to left':



(19)

Initialization:  (identity), Iteration step: 

(20)

<u>Normalization</u> $\langle \psi | \psi \rangle = \,?$

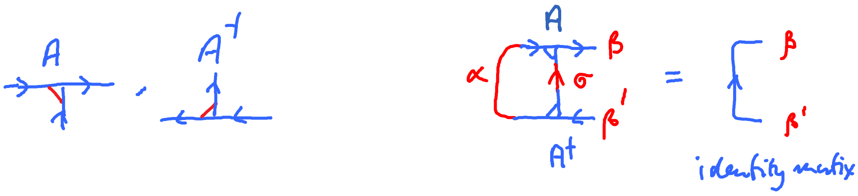
Use above scheme, with $\tilde{A} = A$

<u>Left-normalization</u>

A 3-leg tensor $A^{\alpha\sigma}{}_{\beta}$ is called 'left-normalized' if it satisfies

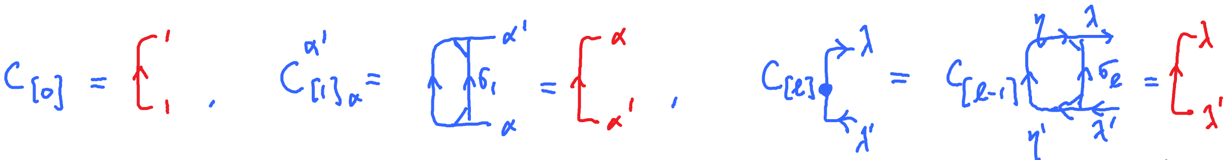
$\boxed{A^{\dagger} A = \mathbb{1}}$. Explicitly: $(A^{\dagger}A)^{\beta'}{}_{\beta} = A^{\dagger}{}^{\beta'}{}_{\sigma\alpha} A^{\alpha\sigma}{}_{\beta} = \mathbb{1}^{\beta'}{}_{\beta}$ (21)

Graphical notation for left-normalization: draw 'left-pointing diagonals' at vertices



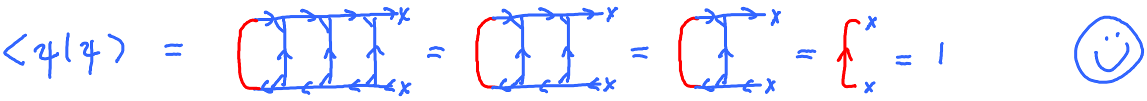
(22)

When all A's are left-normalized, closing the zipper left-to-right is easy, since all $C_{[\ell]}$ reduce to identity matrices:



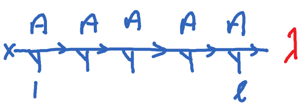
(23)

Hence:

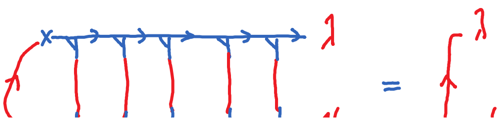


(24)

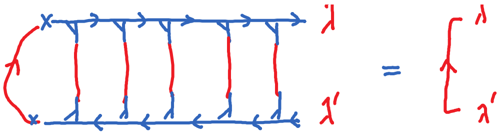
When all matrices of a MPS are left-normalized, the matrices for site 1 to any site $\ell = 1, \dots, N$ define an orthonormal state space:



$|\lambda\rangle = |\vec{\sigma}_\ell\rangle \, [A^{\sigma_1} A^{\sigma_2} \dots A^{\sigma_\ell}]^1{}_{\lambda}$



$\langle \lambda' | \lambda \rangle = \mathbb{1}^{\lambda'}{}_{\lambda}$



close the zipper

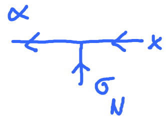
$$\langle\lambda'|\lambda\rangle = \mathbb{1}^{\lambda'}{}_{\lambda}$$

<u>Right-normalization</u>

So far we have viewed an MPS as being built up from left to right, hence used right-pointing arrows on ket diagram. Sometimes it is useful to build it up from right to left, running left-pointing arrows.

Building blocks:

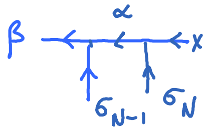
$$|\alpha\rangle = |\sigma_N\rangle B_\alpha{}^{\sigma_N 1}$$



(25)

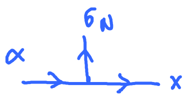
$$|\beta\rangle = |\sigma_N\rangle|\sigma_{N-1}\rangle\, B_\beta{}^{\sigma_{N-1}\alpha} B_\alpha{}^{\sigma_N 1}$$

left-to-right index order as in diagram



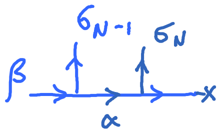
(26)

$$\langle\alpha| = \underbrace{B^\dagger{}_{1\sigma_N}{}^{\alpha}}_{=\overline{B_\alpha{}^{\sigma_N 1}}}\langle\sigma_N|$$



(27)

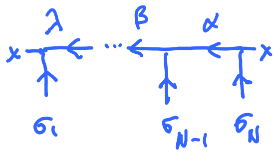
$$\langle\beta| = B^\dagger{}_{1\sigma_N}{}^{\alpha}\, B^\dagger{}_{\alpha\sigma_{N-1}}{}^{\beta}\, \langle\sigma_{N-1}|\langle\sigma_N|$$



(28)

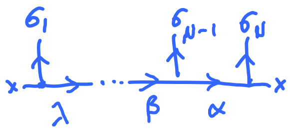
Iterating this, we obtain kets and bras of the form

$$|\psi\rangle = |\sigma_N\rangle|\sigma_{N-1}\rangle\dots|\sigma_1\rangle B_1{}^{\sigma_1\lambda}\dots B_\beta{}^{\sigma_{N-1}\alpha}\, B_\alpha{}^{\sigma_N 1}$$



(29)

$$\langle\psi| = B^\dagger{}_{1\sigma_N}{}^{\alpha}\, B^\dagger{}_{\alpha\sigma_{N-1}}{}^{\beta} \dots B^\dagger{}_{\lambda\sigma_1}{}^{1}\langle\sigma_1|\dots\langle\sigma_{N-1}|\langle\sigma_N|$$

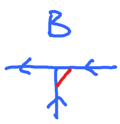


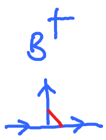
(30)

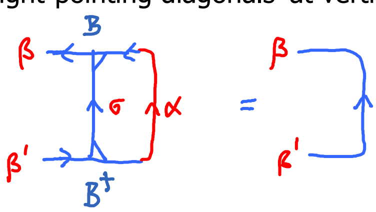
A three-leg terror $B_\beta{}^{\sigma\alpha}$ is called right-normalized if it satisfies

$BB^\dagger = \mathbb{1}$. Explicitly: $(BB^\dagger)_\beta{}^{\beta'} = B_\beta{}^{\sigma\alpha}\, B^\dagger{}_{\alpha\sigma}{}^{\beta'} = \mathbb{1}_\beta{}^{\beta'}$ (31)

Graphical notation for right-normalization: draw 'right-pointing diagonals' at vertices

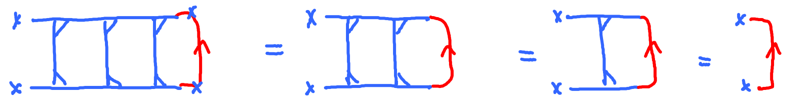




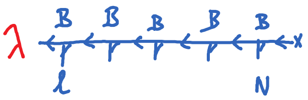


(32)

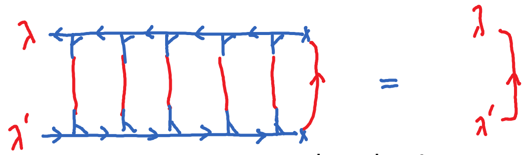
When all B's are right-normalized, closing the zipper right-to-left is easy:

$\langle\psi|\psi\rangle =$  $= 1$ (33)

When all matrices of a MPS are right-normalized, the matrices for site N to any site $\ell = 1, \ldots, N$ define an orthonormal state space:



$$|\lambda\rangle = |\vec{\sigma}_\ell\rangle \left[B^{\sigma_\ell} B^{\sigma_{\ell+1}} \cdots B^{\sigma_N}\right]_{\lambda 1}$$



close the zipper

$$\langle\lambda'|\lambda\rangle = \mathbb{1}^{\lambda'}{}_{\lambda}$$

Conclusion: MPS built purely from left-normalized A's or purely from right-normalized B's are automatically normalized to 1. Shorter MPSs built on subchains automatically define orthonormal state spaces.

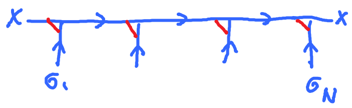
2. Various canonical MPS forms

Any matrix product can be expressed through different matrices without changing the product:

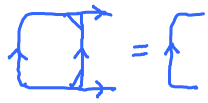
$$M\,M' = \underbrace{(M\,U)}_{\tilde{M}}\underbrace{(U^{-1}M')}_{\tilde{M}'} = \tilde{M}\,\tilde{M}' \qquad \text{'gauge freedom'}$$

Gauge freedom can be exploited to 'reshape' MPSs into particularly convenient, 'canonical' forms:

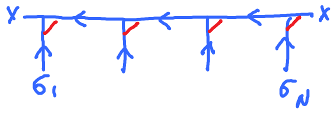
Left-canonical (lc-) MPS:



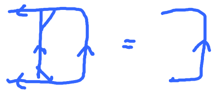
$$|\psi\rangle = |\vec{\sigma}\rangle_N (A^{\sigma_1}\dots A^{\sigma_N}) \qquad A^\dagger A = \mathbb{1} \qquad (1)$$



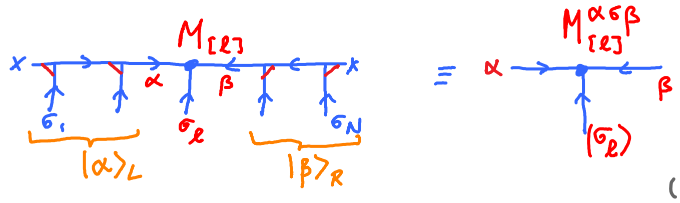
Right-canonical (rc-) MPS:



$$|\psi\rangle = |\vec{\sigma}\rangle_N (B^{\sigma_1}\dots B^{\sigma_N}) \qquad B B^\dagger = \mathbb{1} \qquad (2)$$



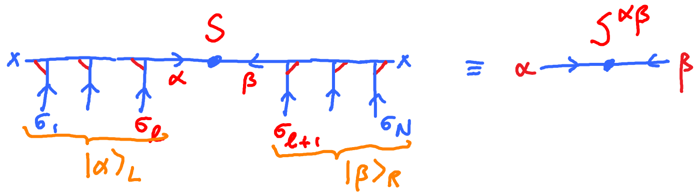
Site-canonical (sc-) MPS:



(3)

$$|\psi\rangle = |\vec{\sigma}\rangle_N (A^{\sigma_1}\dots A^{\sigma_{\ell-1}} M^{\sigma_\ell}_{[\ell]} B^{\sigma_{\ell+1}}\dots B^{\sigma_N}) = |\beta\rangle_R |\sigma_\ell\rangle |\alpha\rangle_L M^{\alpha\sigma_\ell\beta}$$

Bond-canonical (bc-) (or mixed) MPS:



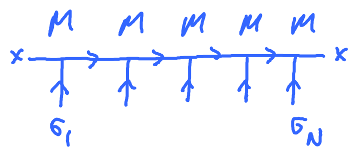
$$|\psi\rangle = |\vec{\sigma}\rangle_N (A^{\sigma_1}\dots A^{\sigma_\ell})_\alpha S^{\alpha\beta}_{[\ell]\beta} (B^{\sigma_{\ell+1}}\dots B^{\sigma_N}) = \sum_{\alpha\beta} |\alpha\rangle_R S^{\alpha\beta}_{[\ell]} |\beta\rangle_L \qquad (4)$$

can be chosen diagonal

How can we bring an arbitrary MPS into one of these forms?

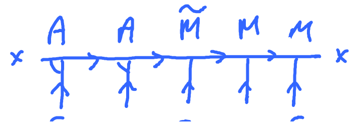
Transforming to left-normalized form

Given: $|\psi\rangle = |\vec{\sigma}\rangle_N (M^{\sigma_1}\dots M^{\sigma_N})$



(5)

[or with index: $|S_N\rangle =$  S_N]



(6)

Goal : left-normalize M^{σ_1} to $M^{\sigma_{\ell-1}}$

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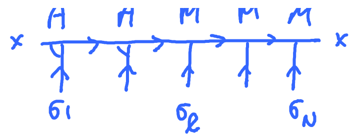
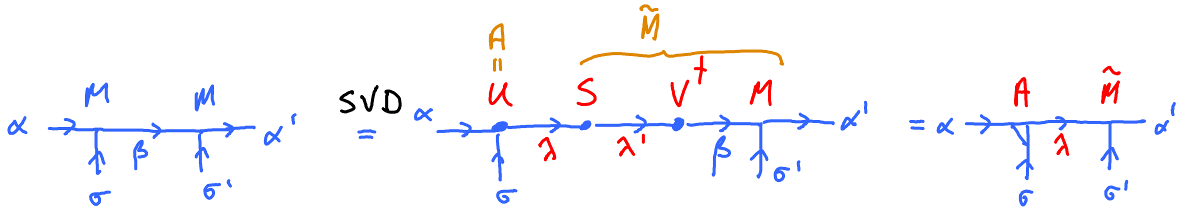


(6)

Strategy: take a pair of adjacent tensors, MM', and use SVD,

$$MM' = USV^\dagger M' \equiv A\tilde{M}, \quad \text{with} \quad A = U, \quad \tilde{M} = SV^\dagger M' \qquad (7)$$



(8)

$$M^{\alpha\sigma}{}_{\beta}\, M^{\beta\sigma'}{}_{\alpha'} = \left(U^{\alpha\sigma}{}_{\lambda}\right)\left(S^{\lambda}{}_{\lambda'}\, V^{\dagger\lambda'}{}_{\beta}\, M^{\beta\sigma'}{}_{\alpha'}\right) = A^{\alpha\sigma}{}_{\lambda}\, \hat{M}^{\lambda\sigma'}{}_{\alpha'} \qquad (9)$$

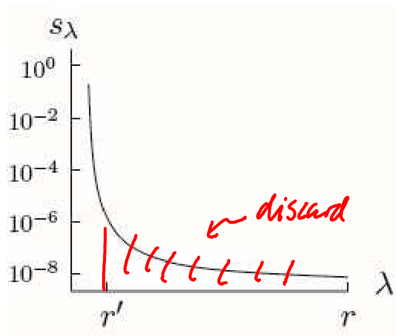
The properly $U^\dagger U = \mathbb{1}$ ensures left-normalization: $A^\dagger A = \mathbb{1}$ (10)

Truncation, if desired, can be performed by discarding some of

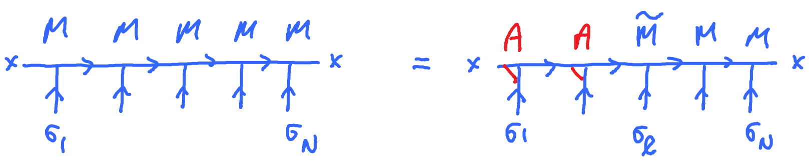
The smallest singular values,

$$\sum_{\lambda=1}^{r} \longrightarrow \sum_{\lambda=1}^{r'} \qquad \text{(but (10) remains valid!)}$$

Note: instead of SVD, we could also me QR (cheaper!)



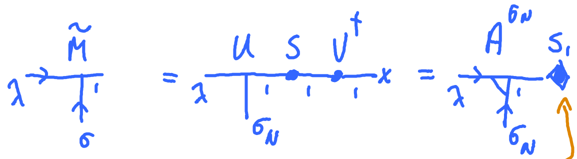
By iterating, starting from $M^{\sigma_1} M^{\sigma_2}$, we left-normalize M^{σ_1} to $M^{\sigma_{\ell-1}}$.



To left-normalize the <u>entire</u> MPS, choose $\ell = N$.

As last step, left-normalize last site using SVD on final $\tilde{M}$:

$$\tilde{M}^{\lambda\sigma_N}{}_{1} = \underbrace{U^{\lambda\sigma_N}{}_{1}}_{A^{\lambda\sigma_N}{}_{1}}\ \underbrace{S^{1}{}_{1}}_{s_1}\ \underbrace{V^{\dagger 1}{}_{1}}_{1}$$



(11)

diamond indicates single number

lc-form: $|\psi\rangle = |\vec{\sigma}\rangle_N \left(A^{\sigma_1} \ldots A^{\sigma_N}\right) s_1$

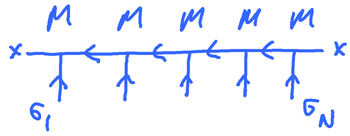
diamond indicates
single number

lc-form: $|\psi\rangle = |\vec{\sigma}\rangle_N \left(A^{\sigma_1} \dots A^{\sigma_N}\right)_{s_1}$

The final singular value, s_1 determines normalization: $\langle\psi|\psi\rangle = |s_1|^2$. (12)

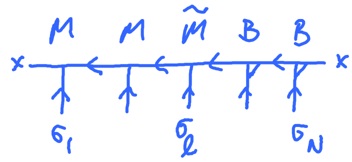
<u>Transforming to right-normalized form</u>

Given: $|\psi\rangle = |\vec{\sigma}\rangle_N \left(M^{\sigma_1} \dots M^{\sigma_N}\right)$



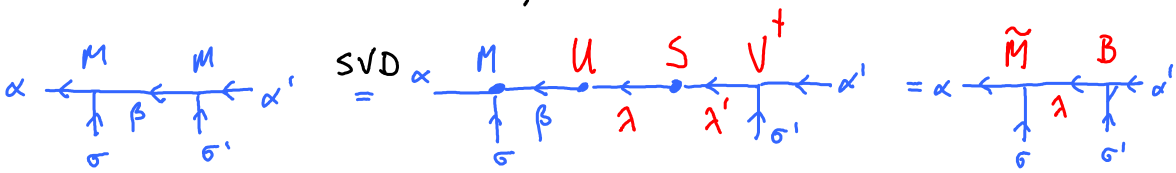
[or with index: $|s_1\rangle = s_1$ ]

Goal : right-normalize M^{σ_N} to $M^{\sigma_{\ell+1}}$



Strategy: take a pair of adjacent tensors, $M M'$, and use SVD:

$$M M' = M U S V^\dagger \equiv \tilde{M} B \,, \quad \text{with} \quad \tilde{M} = M U S \,, \quad B = V^\dagger \,. \qquad (13)$$



(14)

$$M_\alpha{}^{\sigma\beta} M_\beta{}^{\sigma'\alpha'} = \left(M_\alpha{}^{\sigma\beta} U_\beta{}^\lambda S_\lambda{}^{\lambda'}\right)\left(V^\dagger{}_{\lambda'}{}^{\sigma'\alpha'}\right) = \tilde{M}_\alpha{}^{\sigma\lambda'} B_{\lambda'}{}^{\sigma'\alpha'} \qquad (15)$$

Here, $V^\dagger V = \mathbb{1}$ ensures right-normalization: $B B^\dagger = \mathbb{1}$. (16)

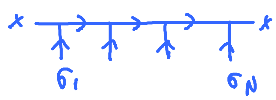
Starting form $M^{\sigma_{N-1}} M^{\sigma_N}$, move leftward up to $M^{\sigma_\ell} M^{\sigma_{\ell+1}}$.

To right-normalize entire chain, choose / and at last site, $\ell = 1$

$$\tilde{M}_1{}^{\sigma_1\lambda} = \underbrace{U_1{}'}_{=1} \underbrace{S_1{}'}_{s_1} \underbrace{V^\dagger{}_1{}^{\sigma_1\lambda}}_{B_1{}^{\sigma_1\lambda}} \,.$$ s_1 determines normalization. (17)

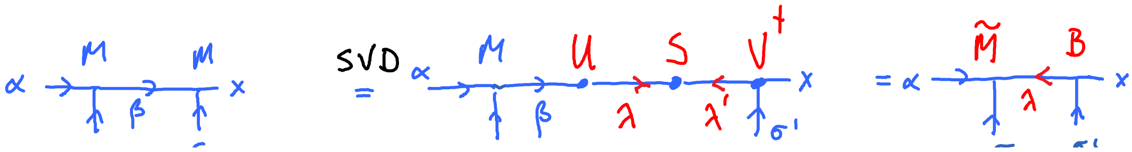
<u>Exercise</u>

(a) Right-normalize a state with right-pointing arrows!

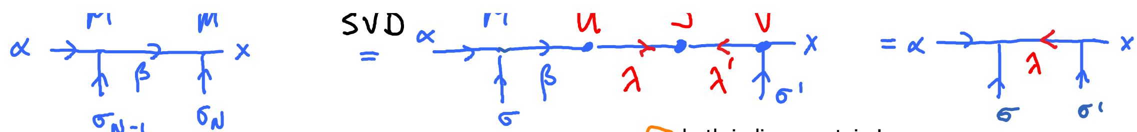


Hint: start at $M^{\sigma_{N-1}} M^{\sigma_N}$

and note the up $\leftrightarrow$ down changes in index placement.



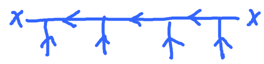
(18)



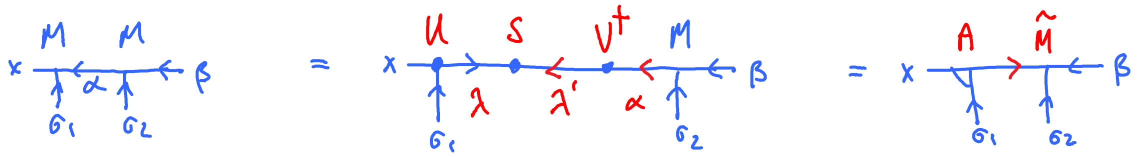
(18)

$$M^{\alpha\sigma}{}_{\beta}\, M^{\beta\sigma'}{}_{1} = \left(M^{\alpha\sigma}{}_{\beta}\, U^{\beta}{}_{\lambda}\, S^{\lambda\lambda'}\right)\left(V^{\dagger}{}_{\lambda'}{}^{\sigma'}\right) = \tilde{M}^{\alpha\sigma\lambda}\, B_{\lambda}{}^{\sigma'} \qquad (19)$$

(b) Left-normalize a state with left-pointing arrows!



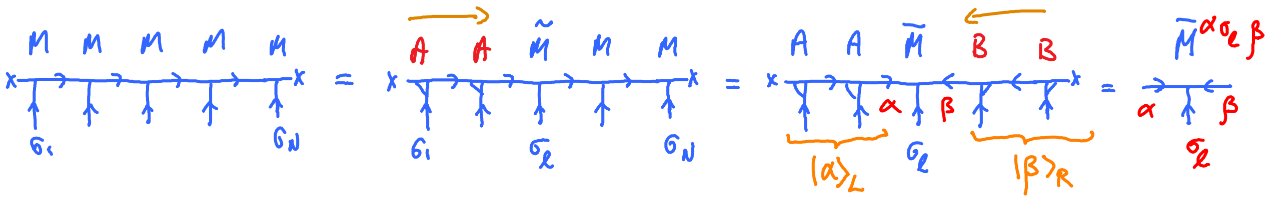
Hint: start at $M^{\sigma_1} M^{\sigma_2}$:



(20)

$$M_{1}{}^{\sigma_1\alpha}\, M_{\alpha}{}^{\sigma_2\beta} = \left(U^{1\sigma_1}{}_{\lambda}\right)\left(S^{\lambda\lambda'}\, V^{\dagger}{}_{\lambda'}{}^{\alpha}\, M_{\alpha}{}^{\sigma_2\beta}\right) = A_{1}{}^{\sigma_1\lambda}\, \tilde{M}_{\lambda}{}^{\sigma_2\beta} \qquad (21)$$

Transforming to site-canonical form



(22)

Left-normalize sites 1 to $\ell-1$, starting from site .

Then right-normalize sites N to $\ell+1$, starting from site N .

Result:

$$|\psi\rangle = \underbrace{|\sigma_N\rangle\dots|\sigma_{\ell+1}\rangle\left(B^{\sigma_{\ell+1}}\dots B^{\sigma_N}\right)^{1}{}_{\beta}}_{|\beta\rangle_R}\, |\sigma_\ell\rangle\, \underbrace{|\sigma_{\ell-1}\rangle\dots|\sigma_1\rangle\left(A^{\sigma_1}\dots A^{\sigma_{\ell-1}}\right)^{1}{}_{\alpha}}_{|\alpha\rangle_L}\, \bar{M}^{\alpha\sigma_\ell\beta} \qquad (23)$$

$$= |\beta\rangle_R\, |\sigma_\ell\rangle\, |\alpha\rangle_L\, \bar{M}^{\alpha\sigma_\ell\beta} \qquad (24)$$

The states $|\alpha,\sigma_\ell,\beta\rangle \equiv |\beta\rangle_R |\sigma_\ell\rangle |\alpha\rangle_L$ form an orthonormal set:

$$\langle \alpha',\sigma_\ell',\beta' | \alpha,\sigma_\ell,\beta\rangle = \delta^{\alpha'}{}_{\alpha}\,\delta^{\sigma_\ell'}{}_{\sigma_\ell}\,\delta^{\beta'}{}_{\beta} \qquad (25)$$

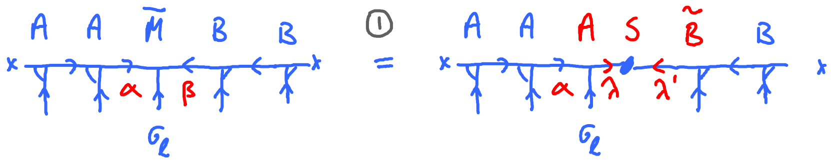
(Exercise: verify this, using $A^\dagger A = \mathbb{1}$ and $BB^\dagger = \mathbb{1}$.)

This is 'local site basis' for site ℓ . Its dimension $D_\alpha \cdot d \cdot D_\beta$ is usually $\lll d^N$ of full Hilbert space.

Transforming to bond-canonical form

Transforming to bond-canonical form

Start from (e.g.) sc-form, use SVD for $\bar{M} = USV^\dagger$, combine ① $V^\dagger$ with neighboring B, or ② U with neighboring A.



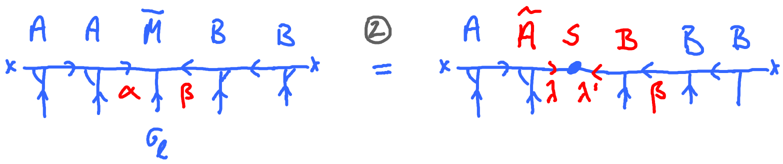
$$= \underbrace{|\lambda'\rangle_R}_{\text{involves sites } \ell+1 \text{ to } N} \cdot \underbrace{|\lambda\rangle_L}_{\text{involves sites } 1 \text{ to } \ell} S^{\lambda\lambda'} \qquad (26)$$

$$\bar{M} = USV^\dagger \qquad A = U, \quad \tilde{B} = V^\dagger B \qquad \text{(Exercise: add indices!)} \qquad (27)$$

The states $|\lambda, \lambda'\rangle \equiv |\lambda\rangle_R |\lambda'\rangle_L$ form an orthonormal set.

$$\langle \bar{\lambda}, \bar{\lambda}' | \lambda, \lambda' \rangle = \delta^{\bar{\lambda}}{}_{\lambda} \, \delta^{\bar{\lambda}'}{}_{\lambda'} \qquad (28)$$

This is called the 'local bond basis for bond ℓ ' (from site ℓ to $\ell+1$). It has dimension $r \cdot r$ (r = dimension of singular matrix S).



$$= \underbrace{|\lambda'\rangle_R}_{\text{involves sites } \ell \text{ to } N} \cdot \underbrace{|\lambda\rangle_L}_{\text{involves sites } 1 \text{ to } \ell-1} S^{\lambda\lambda'} \qquad (29)$$

$$\bar{M} = USV^\dagger \qquad \tilde{A} = AU, \quad B = V^\dagger \qquad \text{(Exercise: add indices!)} \qquad (30)$$

$|\lambda, \lambda'\rangle \equiv |\lambda'\rangle_R |\lambda\rangle_L$ form 'local bond basis' for bond $\ell-1$ (from site $\ell-1$ to ℓ).