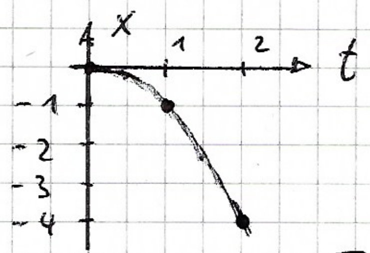
3.4. Skalenverhalten der Bewegungsgleichungen

Skalentrafo: $\vec{r}_a \to \vec{r}_a{}' = \alpha \vec{r}_a$, $t \to t' = \beta t$ $(\square)$

z.B.



$x = -\frac{g}{2} t^2$

$\to x' = -\frac{g}{2} t'^2 \iff \alpha x = \beta^2 \left(-\frac{g}{2}\right) t^2$

Für $\alpha = \beta^2 \to$ reskalierte Lösung d. Bew.gl.

$(\square) \Rightarrow \vec{v}_a = \frac{d\vec{r}_a}{dt} \to \vec{v}_a{}' = \frac{\alpha}{\beta} \vec{v}_a$

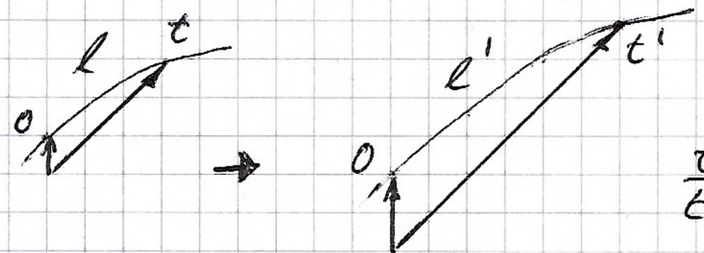
$\Rightarrow$ kin. Energie $T \to T' = \frac{\alpha^2}{\beta^2} T$

Annahme : $U(\vec{r}_a) \to U(\alpha \vec{r}_a) = \alpha^k U(\vec{r}_a)$ homogen, Grad k

$\Rightarrow L = T - U \to L' = T' - U' = \frac{\alpha^2}{\beta^2} T - \alpha^k U \overset{(*)}{=} \alpha^k L$

(*) falls $\frac{\alpha^2}{\beta^2} = \alpha^k$ $\boxed{\beta = \alpha^{1-\frac{k}{2}}}$

$\to$ ELG unverändert, geometrisch ähnliche Bahnen



$\frac{l'}{l} = \alpha$

$\frac{t'}{t} = \beta = \alpha^{1-\frac{k}{2}} = \left(\frac{l'}{l}\right)^{1-\frac{k}{2}}$

$\frac{v'}{v} = \frac{\alpha}{\beta} = \left(\frac{l'}{l}\right)^{k/2}$ $\frac{E'}{E} = \left(\frac{l'}{l}\right)^k$ $\frac{|\vec{L}'|}{|\vec{L}|} = \left(\frac{l'}{l}\right)^{1+\frac{k}{2}}$

a.) $k = 1$ (freier Fall) $\Rightarrow \frac{t'}{t} = \left(\frac{l'}{l}\right)^{1/2}$

b.) $k = 2$ (harmonischer Oszillator) $\Rightarrow \frac{t'}{t} = 1$

c.) $k = -1$ (Gravitation) $\Rightarrow \frac{t'}{t} = \left(\frac{l'}{l}\right)^{3/2}$

3. Keplersches Gesetz

Virialsatz

$$2T = \sum_a \dot{\vec{r}}_a \frac{\partial T}{\partial \dot{\vec{r}}_a} = \sum_a \dot{\vec{r}}_a \vec{p}_a = \frac{d}{dt}\left(\sum_a \vec{r}_a \vec{p}_a\right) - \sum_a \vec{r}_a \dot{\vec{p}}_a$$

zeitliches Mittel: $\quad \bar{f} := \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau f(t)\, dt$

falls $f(t) = \frac{d}{dt} F(t)$, $F(t)$ beschränkt $\forall t$

$$\Rightarrow \bar{f} = \lim_{\tau \to \infty} \frac{F(\tau) - F(0)}{\tau} = 0$$

Falls $\sum \vec{r}_a \vec{p}_a$ beschränkt $\Rightarrow \overline{\frac{d}{dt}\left(\sum \vec{r}_a \vec{p}_a\right)} = 0$

$$\Rightarrow 2\bar{T} = -\overline{\sum \vec{r}_a \dot{\vec{p}}_a} \overset{ELG}{=} \overline{\sum \vec{r}_a \frac{\partial U}{\partial \vec{r}_a}}$$

Für $U(\alpha \vec{r}_a) = \alpha^k U(\vec{r}_a) \Rightarrow \sum \vec{r}_a \frac{\partial U}{\partial \vec{r}_a} = k\, U$ $\quad$ Eulers Theorem

$$\Rightarrow \boxed{2\bar{T} = k\,\bar{U}}$$

$E = T + U$, $\bar{E} = \bar{T} + \bar{U} = E \Rightarrow \bar{U} = \frac{2}{k+2} E$, $\bar{T} = \frac{k}{k+2} E$

harmonischer Oszillator $\quad k = 2 \quad \bar{T} = \bar{U} = \frac{1}{2} E$

Gravitation $\quad k = -1 \quad 2\bar{T} = -\bar{U}$

$E = -\bar{T} < 0 \Leftrightarrow$ gebundenes System