Variationsrechnung

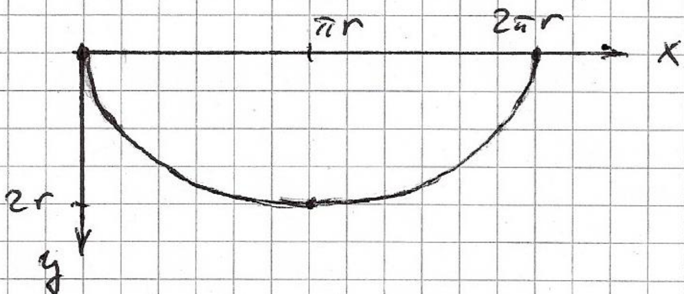
$$F[y(x)] = \int_a^b dx\, f(y, y', x)$$

$$\delta F = 0 \quad \Leftrightarrow \quad \frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \qquad \text{ELG}$$

z.B. Brachistochrone (↑ S. 6)

$$f(y, y', x) = \sqrt{\frac{1+y'^2}{y}} \qquad \text{ELG} \Rightarrow 1 + 2yy'' + y'^2 = 0$$

Lösung: Zykloide



$$y = r(1 - \cos\varphi)$$

$$x = r(\varphi - \sin\varphi)$$

Verallgemeinerung:

mehrere Funktionen (↔ Punktmechanik)

$$F[y_1(x), \ldots, y_n(x)] = \int_a^b dx\, f(y_1, \ldots, y_n; y_1', \ldots, y_n'; x)$$

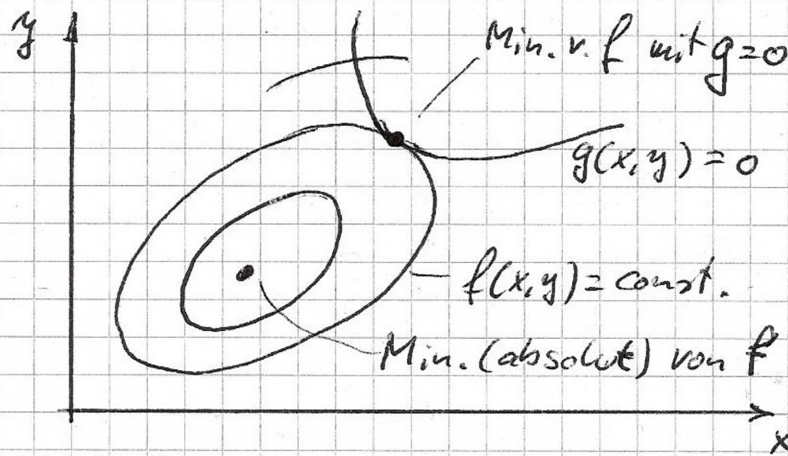
$$\equiv \int_a^b dx\, f(\underline{y}, \underline{y}', x)$$

$$\text{ELG:} \quad \frac{d}{dx}\frac{\partial f}{\partial y_i'} - \frac{\partial f}{\partial y_i} = 0 \qquad i = 1, \ldots, n$$

Variation mit Nebenbedingungen

- gewöhnliche Analysis:

$$\frac{f(x,y) \text{ minimal}}{\text{mit } g(x,y)=0}$$



Methode der Lagrange-Multiplikatoren:

minimiere $\boxed{f(x,y)-\lambda\, g(x,y)}$

$$\xrightarrow{\partial_x,\partial_y,\partial_\lambda \to 0} \quad \frac{\partial f}{\partial x}-\lambda\frac{\partial g}{\partial x}=0, \quad \frac{\partial f}{\partial y}-\lambda\frac{\partial g}{\partial y}=0, \quad g(x,y)=0$$

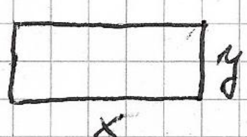
$$\Rightarrow \begin{pmatrix}\partial f/\partial x\\ \partial f/\partial y\end{pmatrix}=\lambda\begin{pmatrix}\partial g/\partial x\\ \partial g/\partial y\end{pmatrix}$$ Gradienten von f und g proportional

$\rightarrow$ Höhenlinien von f, g tangential, mit $g=0$

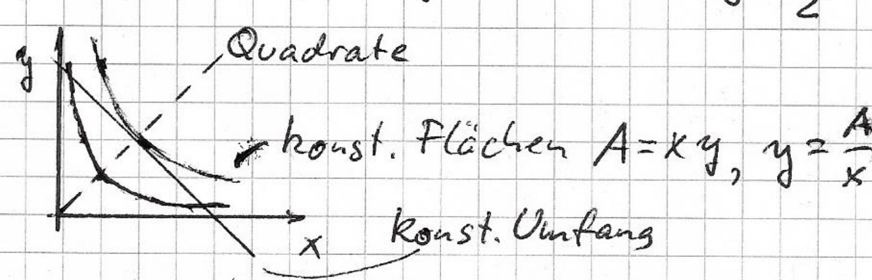
$\hat{=}$ Extremum von f unter Bedingung $g(x,y)=0$ ☑

Beispiel: Rechteck mit gegebenem Umfang $2L$ maximale Fläche?

maximiere $xy-\lambda(x+y-L)$



$$\Rightarrow y-\lambda=0, \quad x-\lambda=0, \quad x+y=L \Rightarrow x=y=\frac{L}{2}$$



- Variationsrechnung

1. isoperimetrische Nebenbedingung

$$F[y(x)] = \int_a^b dx\, f(y, y', x) \qquad G[y(x)] = \int_a^b dx\, g(y, y', x) = c$$

$y(a)$, $y(b)$ fest

<u>eine</u> Bedingung

2. holonome Nebenbedingung

$$F[y_i(x)] = \int_a^b dx\, f(y_i, y_i', x) \qquad g(y_i, x) = 0$$

∞ viele Bed. $(\forall x)$

---

zu 1.: Extremum von

$$F[y(x)] - \lambda\left(G[y(x)] - c\right) \longrightarrow \frac{d}{dx}\left(\frac{\partial f}{\partial y'} - \lambda \frac{\partial g}{\partial y'}\right) = \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y}$$

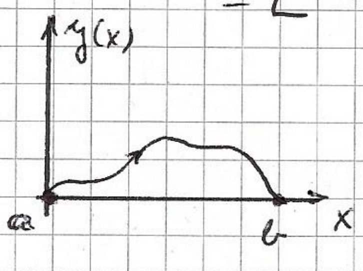
<u>Beispiel:</u>

max. Fläche $F[y(x)] = \int_a^b dx\, y(x)$, konst. Kurvenlänge $G[y] = \int_a^b dx \sqrt{1+y'^2} = L$

ELG:

$$\frac{d}{dx}\left(-\lambda \frac{y'}{\sqrt{1+y'^2}}\right) = 1 \Rightarrow -\lambda \frac{y'}{\sqrt{1+y'^2}} = x - c_1$$

$$\Rightarrow y(x) - b_1 = \sqrt{\lambda^2 - (x-c_1)^2} \quad \text{<u>Kreis</u>}$$



zu 2.: Extremum von $F^*[y_i(x), \lambda(x)] = \int_a^b dx\left(f(y_i, y_i', x) - \lambda(x)\, g(y_i, x)\right)$

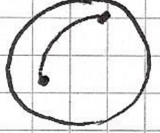
ELG: $\dfrac{d}{dx}\dfrac{\partial f}{\partial y_i'} = \dfrac{\partial f}{\partial y_i} - \lambda(x)\dfrac{\partial g}{\partial y_i}$ $\quad i = 1, \ldots, n \qquad g(y_i, x) = 0$

<u>Beispiel:</u> geodätische Linien

Kurvenlänge $F[x(t), y(t), z(t)] = \int_{t_1}^{t_2} dt \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

mit Bedingung $\quad g(x, y, z) = 0 \quad \to$ 2-dim. Fläche in $\mathbb{R}^3$

z.B. $\quad g(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0$



Kugeloberfläche

2.2. Das Prinzip der kleinsten Wirkung (Hamiltons Prinzip)

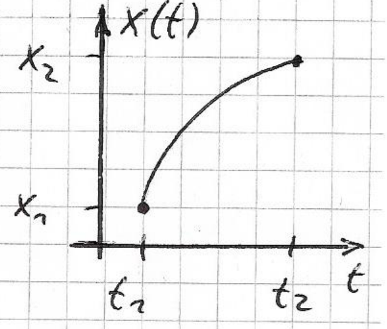
eindimensionale Bewegung $x(t)$

Potential $U(x)$, kin. Energie $T = \frac{m}{2}\dot{x}^2$

Def.: Lagrangefunktion:

$$L = L(x,\dot{x},t) = T - U \qquad \text{hier } L = L(x,\dot{x})$$

Wirkung: $S = S[x(t)] = \int_{t_1}^{t_2} dt\, L(x,\dot{x},t)$



Die tatsächliche Bewegung $x(t)$ zwischen $x_1 = x(t_1)$ und $x_2 = x(t_2)$ verläuft so, dass S minimal wird:

$$\boxed{\delta S = 0}$$

$$L = \frac{m}{2}\dot{x}^2 - U(x)$$

$$\delta S = 0 \iff \text{ELG} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \iff m\ddot{x} = -\frac{\partial U}{\partial x}$$

Variationsprinzip — Newtons 2. Axiom

Randbed.: $\underline{x(t_1),\, x(t_2)}$ — $\underline{x(t_1),\, \dot{x}(t_1)}$