2. Lagrangeformalismus und das Prinzip der kleinsten Wirkung

2.1 Einführung in die Variationsrechnung

Funktion $\quad y \in \mathbb{R} \longrightarrow F(y)$

Fkt. mehrerer Variabler $\quad \vec{y} \in \mathbb{R}^n \longrightarrow F(\vec{y}) \equiv F(y_i),\ i=1,..,n$

z.B. $\quad F(y_i) = y_1^2 + y_2^2 + \ldots + y_n^2 = |\vec{y}|^2$

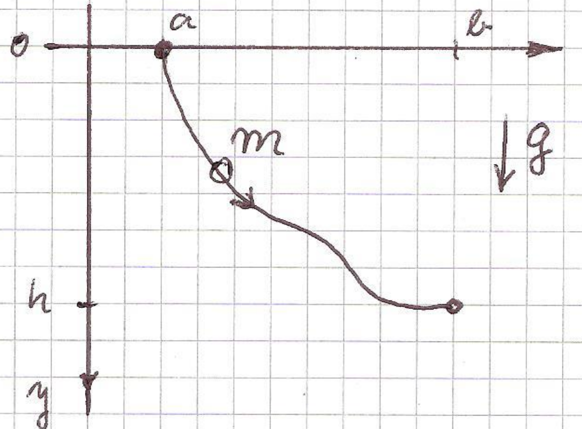
<u>Funktional</u> $\quad y(x) \longrightarrow F[y(x)] \qquad x \in [a,b]$

Analogie: $\quad y_i \longrightarrow F(y_i) \qquad i=1,\ldots,n$

Fkt. $y(x)$ $\hat{=}$ "unendlichdimensionaler Vektor", kontinuierlicher Index x

Beispiel: $\quad F[y(x)] = \int_a^b dx\ y^2(x)$

Problem der <u>Brachistochrone</u> Johann Bernoulli 1696



$$T = F[y(x)] = \int \frac{ds}{v}$$

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1+y'^2}$$

$$y' = \frac{dy}{dx}$$

$$\frac{m}{2} v^2 = m g y \Rightarrow v = \sqrt{2 g y(x)}$$

$$F[y(x)] = \frac{1}{\sqrt{2g}} \int_a^b dx \sqrt{\frac{1+y'(x)^2}{y(x)}}$$

Minimum von F als Funktional von $y(x)$?

allg.: $F[y(x)] = \int_a^b dx\, f(y(x), y'(x), x) \equiv \int_a^b dx\, f(y, y', x)$

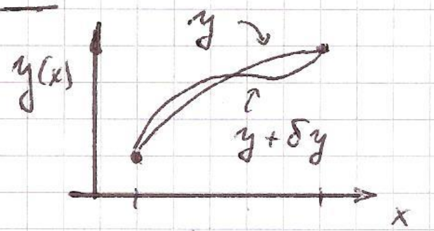
Minimum (Extremum) von $F[y(x)]$? $y(a), y(b)$ fest

$\Leftrightarrow$ <u>Funktionalanalysis</u>

normale Funktion $F(y_i) = \sum_i y_i^2$

Min. v. F: $dF = F(y_i + dy_i) - F(y_i) = \sum_i \frac{\partial F}{\partial y_i} dy_i \overset{!}{=} 0 \quad \forall dy_i$

$\Rightarrow \frac{\partial F}{\partial y_i} = 0$; z.B. $\frac{\partial F}{\partial y_i} = 2y_i \overset{!}{=} 0 \Leftrightarrow \vec{y} = 0$



erste Variation des Funktionals

$\delta F = F[y(x) + \delta y(x)] - F[y(x)] =$

$= \int_a^b dx\, f(y+\delta y, y'+\delta y', x) - f(y, y', x)$

<u>δy infinitesimal</u>

$\doteq \int_a^b dx \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right)$

$\delta y' \equiv \tilde{y}' - y' = \frac{d}{dx}(\tilde{y} - y) = \frac{d}{dx} \delta y$

$= \int_a^b dx \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \frac{d}{dx} \delta y \right)$

$\delta y(a) = \delta y(b) = 0$

$\overset{\text{part.Int.}}{=} \int_a^b dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \cdot \delta y(x) \overset{!}{=} 0 \quad \forall \delta y(x)$

$\Leftrightarrow$ <u>Euler-Lagrange-Gleichungen</u>

$$\boxed{\frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0}$$

DGL 2. Ordnung für $y(x)$ linear in y''

Bedingung für Extremum von $F[y(x)] = \int_a^b dx\, f(y, y', x)$
(F stationär)

Präsenzübung 1

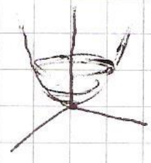
1. Minimiere:

a.) $F(\vec{y}) = y_1^2 + y_2^2$

b.) $F[y] = \int_a^b dx\, y^2(x)$, $y(a) = y(b) = 0$

c.) $F[y] = \int_a^b dx\, (y'(x))^2$

a.) $\frac{\partial F}{\partial y_1} = 2y_1 \overset{!}{=} 0$, $\frac{\partial F}{\partial y_2} = 2y_2 \overset{!}{=} 0 \Rightarrow \vec{y} = 0$



b.) $f = y^2$, $\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \Leftrightarrow \frac{\partial f}{\partial y} = 2y = 0 \quad y(x) \equiv 0$

oder: $\delta F = \int_a^b dx\, 2y\, \delta y = 0, \ \forall \delta y \Rightarrow y(x) = 0$

c.) $\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \Leftrightarrow \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \Leftrightarrow \frac{\partial f}{\partial y'} = 2y' = 2C_1$

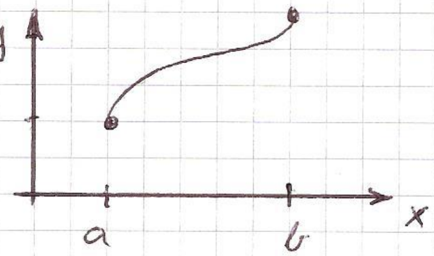
$\Rightarrow y = C_1 x + C_2$

2. $F[y] = \int_a^b dx\, f(y, y', x)$ stationär

es sei $\frac{\partial f}{\partial x} = 0$, d.h. $f = f(y, y')$ zeige: $y'\frac{\partial f}{\partial y'} - f = \text{const.}$

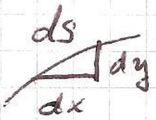
$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'} - f\right) = y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\cdot y' - \frac{\partial f}{\partial y'}\cdot y'' \overset{ELG}{=} 0$$

✓

3. 

Kürzeste Verbindung?

$y(a) = y_1 \quad , \quad y(b) = y_2$



Streckenlänge $\quad ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\, dx$

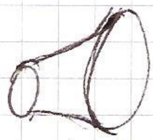
$$S = F[y] = \int_a^b \sqrt{1+y'^2}\, dx \quad \longrightarrow \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} = \text{const.}$$

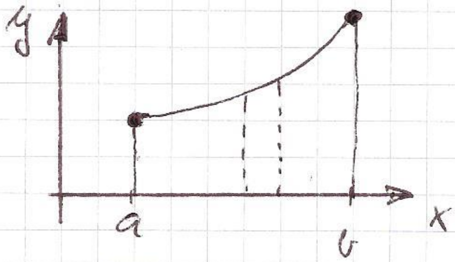
$$\Rightarrow y' = c_1 = \text{const.} \qquad \underline{y = c_1 x + c_2}$$

$$y(a) = c_1 a + c_2 = y_1 \quad , \quad c_1 b + c_2 = y_2$$

$$\Rightarrow \quad c_1 = \frac{1}{b-a}(y_2 - y_1) \qquad c_2 = \frac{1}{b-a}(b y_1 - a y_2)$$

4. Minimalfläche





$$F[y] = 2\pi \int_a^b y \sqrt{1+y'^2}\, dx$$

$$2. \Rightarrow \quad y' \frac{\partial f}{\partial y'} - f = y \frac{y'^2}{\sqrt{1+y'^2}} - y \frac{1+y'^2}{\sqrt{1+y'^2}} = \frac{-y}{\sqrt{1+y'^2}} = c_1$$

$$\Rightarrow 1 + y'^2 = \left(\frac{y}{c_1}\right)^2 \quad \Rightarrow y' = \sqrt{\left(\frac{y}{c_1}\right)^2 - 1} \quad \Rightarrow dx = \frac{dy}{\sqrt{\left(\frac{y}{c_1}\right)^2 - 1}}$$

$$\Rightarrow x - c_2 = c_1 \ln\left(\frac{y}{c_1} + \sqrt{\frac{y^2}{c_1^2} - 1}\right) \quad \Leftrightarrow \quad \frac{y}{c_1} = \frac{1}{2}\left(e^{\frac{x-c_2}{c_1}} + e^{-\frac{x-c_2}{c_1}}\right)$$