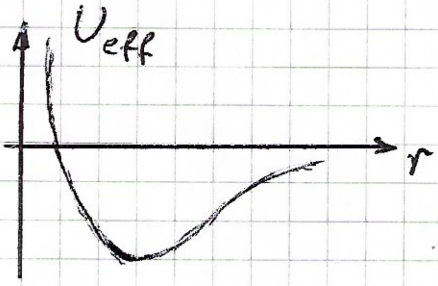
# 4.3. Kepler-Problem




$$U(r) = -\frac{\alpha}{r} \qquad \alpha = G m_1 m_2 > 0$$

$$U_{eff}(r) = -\frac{\alpha}{r} + \frac{\ell^2}{2mr^2}$$

Bahn:

$$\varphi = \underbrace{\pm \int \frac{\frac{\ell}{r^2}\, dr}{\sqrt{2m(E-U(r)) - \frac{\ell^2}{r^2}}}}_{\equiv J} + \text{const.}$$

$$u = \frac{1}{r} \qquad du = -\frac{dr}{r^2}$$

$$J = -\int \frac{\ell\, du}{\sqrt{2m(E-U(\frac{1}{u})) - \ell^2 u^2}} = -\int \frac{du}{\sqrt{\frac{2mE}{\ell^2} + \frac{2m\alpha u}{\ell^2} - u^2}} =$$

$$= -\int \frac{du}{\sqrt{C_1 + C_2 u + C_3 u^2}} \qquad C_1 = \frac{2mE}{\ell^2}, \quad C_2 = \frac{2m\alpha}{\ell^2}, \quad C_3 = -1$$

$$= \frac{1}{\sqrt{-C_3}} \arccos\left[-\frac{C_2 + 2C_3 u}{\sqrt{C_2^2 - 4C_1C_3}}\right] \quad ; \quad \frac{d \arccos x}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$= \arccos\left[-\frac{\frac{\cancel{2}m\alpha}{\ell^2} - \frac{\cancel{2}}{r}}{\sqrt{\frac{\cancel{4}m^2\alpha^2}{\ell^4} + \cancel{4}\frac{2mE}{\ell^2}}}\right] = \arccos \frac{\frac{\ell^2}{m\alpha r} - 1}{\sqrt{1 + \frac{2E\ell^2}{m\alpha^2}}} =$$

$$= \arccos\left(\frac{\frac{p}{r} - 1}{e}\right) \qquad p = \frac{\ell^2}{m\alpha} \qquad e = \sqrt{1 + \frac{2E\ell^2}{m\alpha^2}}$$

Parameter, Exzentrizität der Bahn

$$\Rightarrow \quad \boxed{\frac{p}{r} = 1 + e\cos\varphi}$$

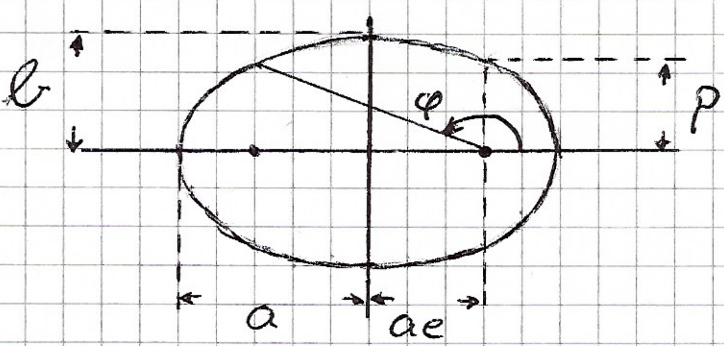
- Kegelschnitt, Brennpunkt im Ursprung
- Integrationskonst.: $\varphi = 0 \leftrightarrow r = r_{min}$ <u>Perihel</u>

| | | |
|---|---|---|
| $E > 0$ | $e > 1$ | Hyperbel |
| $E = 0$ | $e = 1$ | Parabel |
| $E < 0$ | $e < 1$ | Ellipse |
| $E = (U_{eff})_{min}$ | $e = 0$ | Kreis |

<u>$e = 0$:</u> $U_{eff} = -\frac{\alpha}{r} + \frac{\ell^2}{2mr^2}$, $\frac{dU_{eff}}{dr} = \frac{\alpha}{r^2} - \frac{\ell^2}{mr^3} \overset{!}{=} 0 \Rightarrow r = \frac{\ell^2}{m\alpha} \equiv p$

$\Rightarrow U_{eff}(p) = -\frac{m\alpha^2}{2\ell^2} \Rightarrow e = 0$ <u>$r = p = \text{const.}$</u>

<u>$e < 1$:</u> Ellipse




$$r(\varphi) = \frac{p}{1 + e\cos\varphi}$$

1. Keplersches Gesetz

$r(0) = \frac{p}{1+e} = a(1-e) = r_{min}$, $r(\pi) = \frac{p}{1-e} = a(1+e) = r_{max}$

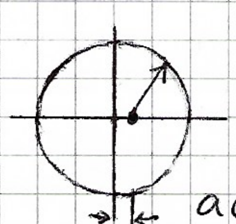
Perihel Aphel

$r(\frac{\pi}{2}) = p$

$a = \frac{p}{1-e^2} = \frac{\alpha}{2|E|}$ <u>unabh. von $\ell$</u> $b = \frac{p}{\sqrt{1-e^2}} = \frac{\ell}{\sqrt{2m|E|}}$

Für $e \ll 1$ gilt bis auf $\mathcal{O}(e^2)$ $a = b = p$

aber: Abstand[Kraftzentrum - Kreismittelpunkt] $= ae$



$$\Delta\varphi = 2\int_{r_{min}}^{r_{max}} f(r)\,dr = 2\left[\arccos\left(\frac{\frac{p}{r}-1}{e}\right)\right]_{r_{min}}^{r_{max}} =$$

$$= 2\left[\arccos\frac{-e}{e} - \arccos\frac{e}{e}\right] = 2\pi \qquad \text{geschlossene Bahn } \checkmark$$

Umlaufzeit $T$: Flächengeschw. $\dot{A} = \frac{\ell}{2m} = \text{const.}$

$$\Rightarrow \quad A = \frac{\ell}{2m}T \qquad \text{Ellipse } A = \pi a b$$

$$\Rightarrow \quad T = \frac{2m\pi a b}{\ell} = \frac{2m\pi}{\ell}\cdot\frac{\alpha}{2|E|}\cdot\frac{\ell}{\sqrt{2m|E|}} = \pi\alpha\sqrt{\frac{m}{2|E|^3}} =$$

$$= 2\pi\sqrt{\frac{m}{\alpha}}\,a^{3/2} \qquad T^2 \sim a^3 \quad \text{3. Keplersches Gesetz}$$

$T(E,\ell) = T(E)$ $\ell$-unabh.

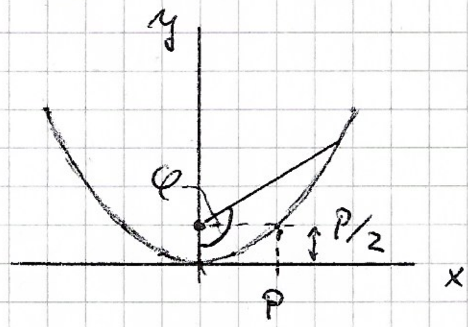
2-Körper-Bewegung: $\vec{R} = 0$, $\vec{r}_1 = \frac{m_2}{m_1+m_2}\vec{r}$, $\vec{r}_2 = -\frac{m_1}{m_1+m_2}\vec{r}$

ebenfalls Ellipsen; $m_2 \gg m_1 \Rightarrow \vec{r}_1 = \vec{r}$, $\vec{r}_2 \to 0$

<u>$e = 1$:</u> Parabel $r = \frac{p}{1+\cos\varphi}$

$$\left.\begin{aligned} x &= r\sin\varphi \\ y &= \frac{p}{2} - r\cos\varphi \end{aligned}\right\} \Rightarrow y = \frac{x^2}{2p}$$

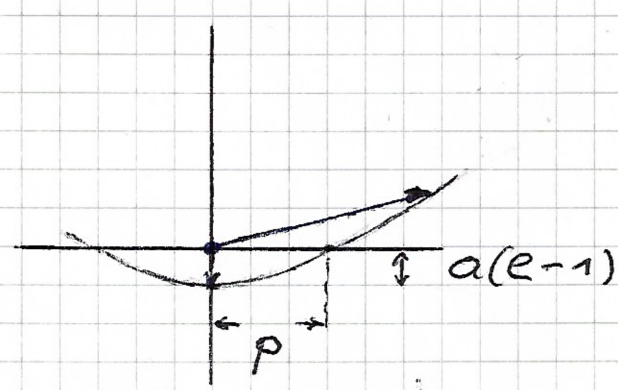
$r_{min} = \frac{p}{2}$ $\qquad r_{max} \to \infty$




<u>$e > 1$:</u> Hyperbel

$$r_{min} = \frac{p}{1+e} = a(e-1)$$

$$a = \frac{p}{e^2-1} = \frac{\alpha}{2E}$$



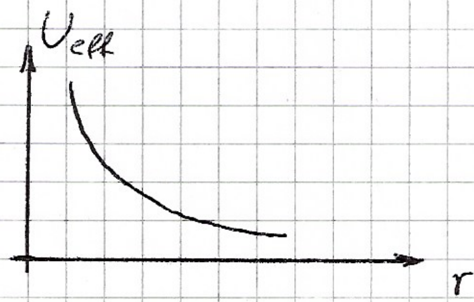
- Ellipse, $E<0$: Zeitabhängigkeit

$$r = a(1 - e\cos\xi) \qquad t = \sqrt{\frac{m a^3}{\alpha}}(\xi - e\sin\xi)$$

$$0 < \xi < 2\pi \quad \Leftrightarrow \quad 0 < t < T$$

$$\cos\varphi = \frac{1}{e}\left(\frac{p}{a(1-e\cos\xi)} - 1\right)$$

- abstoßendes Potential




$$U = -\frac{\alpha}{r}, \quad \alpha < 0$$

$$\frac{p}{r} = -1 + e\cos\varphi \qquad p = \frac{\ell^2}{m|\alpha|}$$

nur $E > 0$ erlaubt, $e = \sqrt{1 + \frac{2E\ell^2}{m\alpha^2}} > 1$, Hyperbel

- <u>Lenz-Runge-Vektor</u>

$$\vec{Q} := \vec{p} \times \vec{\ell} - m\alpha\frac{\vec{r}}{r} \qquad \text{Perihel-Richtung}$$

Erhaltungsgröße, $\frac{d}{dt}\vec{Q} = 0$ $\qquad \vec{\ell} = \vec{r} \times \vec{p}$

$\rightarrow \vec{\ell} \cdot \vec{Q} = 0, \quad \vec{Q}^2 = m^2\alpha^2 e^2 \qquad (*)$

unabh. Erhaltungsgrößen: $2s - 1 = 5$ für $s = 3$ $(r_x, r_y, r_z)$

$$E, \vec{\ell}, \vec{Q} : 7 \xrightarrow{\text{2 Bedingungen } (*)} 5 \quad \checkmark$$