Kanonischer Formalismus, Phasenraum

1. Harmonischer Oszillator $H(p,q) = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$

kanonische Trafo mit Erzeugender $F(q,Q) = \frac{m}{2}\omega q^2 \cot Q$

$$\Rightarrow p = \frac{\partial F}{\partial q} = m\omega q \cot Q \quad \underline{(1)}, \quad \mathcal{P} = -\frac{\partial F}{\partial Q} = \frac{m}{2}\omega q^2 \frac{1}{\sin^2 Q} \quad \underline{(2)}$$

a.) Zeige: $q(\mathcal{P},Q) = \sqrt{\frac{2\mathcal{P}}{m\omega}} \sin Q$, $p(\mathcal{P},Q) = \sqrt{2m\omega\mathcal{P}} \cos Q$

$$(2) \Rightarrow 2m\omega\mathcal{P} = (m\omega q)^2 \frac{1}{\sin^2 Q} \Rightarrow m\omega q = \sqrt{2m\omega\mathcal{P}} \sin Q \Rightarrow \text{Beh}$$

b.) Zeige:
$$Q(p,q) = \arctan \frac{m\omega q}{p}$$
$$\mathcal{P}(p,q) = \frac{p^2}{2m\omega} + \frac{m\omega}{2} q^2 = \frac{H}{\omega}$$

$$a.) \Rightarrow \frac{m\omega q}{p} = \tan Q \Rightarrow Q = \arctan \frac{m\omega q}{p}$$

$$\frac{p^2}{2m\omega} + \frac{m\omega}{2} q^2 \overset{a.)}{=} \mathcal{P}\cos^2 Q + \mathcal{P}\sin^2 Q = \mathcal{P} \quad \square$$

c.) Berechne $D = \frac{\partial(Q,\mathcal{P})}{\partial(q,p)}$

$$\frac{\partial(Q,\mathcal{P})}{\partial(q,p)} = \begin{vmatrix} \partial Q/\partial q & \partial Q/\partial p \\ \partial \mathcal{P}/\partial q & \partial \mathcal{P}/\partial p \end{vmatrix} = \begin{vmatrix} \dfrac{\frac{m\omega}{p}}{1+\left(\frac{m\omega q}{p}\right)^2} & \dfrac{-\frac{m\omega q}{p^2}}{1+\left(\frac{m\omega q}{p}\right)^2} \\ m\omega q & \frac{p}{m\omega} \end{vmatrix} =$$

$$= \frac{1}{1+\left(\frac{m\omega q}{p}\right)^2} + \frac{\left(\frac{m\omega q}{p}\right)^2}{1+\left(\frac{m\omega q}{p}\right)^2} \equiv 1 \quad \checkmark$$

d.,

$$b.) \Rightarrow p(\mathcal{P}, q) = \sqrt{2m\omega\mathcal{P} - m^2\omega^2 q^2}$$

$$a.) \Rightarrow Q(\mathcal{P}, q) = \arcsin \sqrt{\frac{m\omega}{2\mathcal{P}}}\, q$$

Verifiziere

$$D = \frac{\partial(Q,P)}{\partial(q,p)} = \frac{\partial(Q,P)}{\partial(q,\mathcal{P})} \Big/ \frac{\partial(q,p)}{\partial(q,\mathcal{P})} = \left.\frac{\partial Q}{\partial q}\right|_{\mathcal{P}} \left[\left.\frac{\partial p}{\partial \mathcal{P}}\right|_q\right]^{-1} = 1$$

$$\left.\frac{\partial Q}{\partial q}\right|_{\mathcal{P}} = \frac{\sqrt{\frac{m\omega}{2\mathcal{P}}}}{\sqrt{1 - \frac{m\omega}{2\mathcal{P}} q^2}} = \frac{\sqrt{m\omega}}{\sqrt{2\mathcal{P} - m\omega q^2}}$$

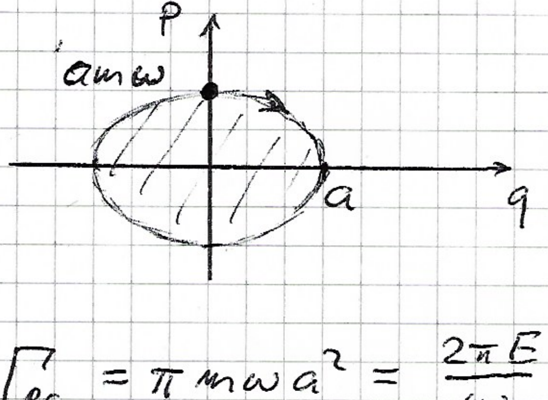
$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

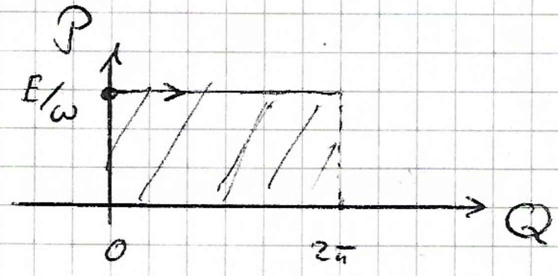
$$\left.\frac{\partial p}{\partial \mathcal{P}}\right|_q = \frac{\cancel{2}\, m\omega}{\cancel{2}\sqrt{2m\omega\mathcal{P} - m^2\omega^2 q^2}} = \left.\frac{\partial Q}{\partial q}\right|_{\mathcal{P}} \Rightarrow D = 1 \quad ✓$$

e., Phasenraum in $p, q \leftrightarrow \mathcal{P}, Q$ $\qquad \mathcal{P} = \frac{E}{\omega} = \text{const.}$

$q = a \sin\omega t \qquad p = am\omega \cos\omega t \qquad Q = \omega t + \alpha = \omega t$

$$a = \sqrt{\frac{2E}{m\omega^2}}$$





$$\Gamma_{pq} = \pi m\omega a^2 = \frac{2\pi E}{\omega} \quad \leftrightarrow \quad \Gamma_{\mathcal{P}Q} = 2\pi \frac{E}{\omega}$$